RATIONAL HOMOTOPY TYPE OF MANIFOLDS

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Abstract

I provide a detailed proof of the rational surgery existence theorem, in both the simply-connected and non-simply-connected case. As applications of the simply-connected case, I study (1) rational homotopy complex projective spaces in terms of their possible Pontryagin numbers. (2) rational analogs of projective planes, which are smooth closed $4k$ dimensional manifolds whose rational cohomology is rank 1 in dimension 0, $2k$ and $4k$ and is zero otherwise. I prove that after dimension 2,4,8 and 16, which are the dimension of the real, complex, quaternionic and octonionic projective planes, 32 is the smallest next dimension where such manifolds do exist. Applying the rational surgery existence theorem, the question is reduced to finding possible Pontryagin classes satisfying the Hirzebruch signature formula and a number of congruence relations determined by the integrality conditions coming from the Riemann-Roch Theorem. And this is eventually equivalent to finding possible solutions to a system of Diophantine equations. As an application of the non-simply-connected case, I study the following question: given a special family of rational Poincaré duality algebras and a finite group action on it, does there exist a free action of the finite group on a smooth closed manifold whose cohomology ring realizes the given algebra with the action? In the last chapter, I study rational surgery existence question in the case when the fundamental group is $\mathbb{Z}$.

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CHAPTER 1

Introduction

1.1. Background

In 1951, Serre proved that the homotopy groups of spheres are all finite except for $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, i.e.

$$\pi_*(S^{2k-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = 2k-1; \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_*(S^{2k}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = 2k \text{ or } 4k-1 \ ; \\ 0 & \text{otherwise} \end{cases}$$

which are fairly simple compared to the very complicated ordinary homotopy groups of spheres. Philosophically, this indicated that one can hope for complete answers to the rational version of certain homotopy theoretic problems that are almost impossible over $\mathbb{Z}$. In the mid 1960's, Sullivan computed the homotopy groups of the classifying space G/PL and introduced the idea of localization at primes, motivated by such type of computations. In his 1970 lecture notes [**S2**], Sullivan initiated the concept of localization and completion of topological spaces. For simply-connected spaces, the localization can be inductively constructed from the Postnikov tower. In 1971, Bousfield and Kan [**BK**] introduced the fibrewise completion and localization functor, which allows a generalization to non-simply-connected spaces. In particular, the localization at (0) associates to a space X a new space $X_{(0)}$ whose higher homotopy groups are all rational vector spaces, and a map $f : X \to X_{(0)}$ inducing an isomorphism on the fundamental group and isomorphisms on higher homotopy groups tensoring the rationals. In [**S1**], Sullivan studied the diffeomorphism classes of compact smooth manifolds determined by algebraic invariants consisting of the rational homotopy type (the minimal model) and the tangent bundle information (including the

Pontryagin classes). He also gave an existence theorem for manifolds realizing such rational homotopy data, followed by a sketch of a proof. In [**TW**], the authors proved the local surgery exact sequence.

1.2. Localization of topological spaces

The localization functor gives a topological space X a CW complex denoted $X_{(0)}$ which carries all the rational homotopy data of the space X.

Definition 1.2.1. A $\mathbb{Q}$-*local space* is a CW complex X, satisfying

$$\pi_*(X) \xrightarrow{\cong} \pi_*(X) \otimes \mathbb{Q} \text{ for all } * > 1.$$

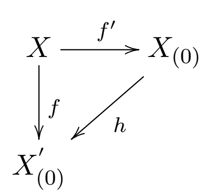
Remark 1.2.2. [**GM**] X is a $\mathbb{Q}$-local space if and only if $\widetilde{H}_*(\widetilde{X}; \mathbb{Z})$ are $\mathbb{Q}$ vector spaces, where $\widetilde{X}$ is the universal cover of X.

Definition 1.2.3. We say a map $f : X \to Y$ is a $\mathbb{Q}$-*homotopy equivalence* if

$$f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(Y) \text{ and } f_* : \pi_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(Y) \otimes \mathbb{Q} \text{ for all } * > 1.$$

Definition 1.2.4. Given a space X and a $\mathbb{Q}$-homotopy equivalence $f : X \to X_{(0)}$ with $X_{(0)}$ $\mathbb{Q}$-local, we say that $f : X \to X_{(0)}$ is a *localization* of X.

Theorem 1.2.5 ([**GM**]). *If $f : X \to X_{(0)}$ and $f' : X \to X'_{(0)}$ are two localizations of X, then there exists a homotopy equivalence $h : X_{(0)} \to X'_{(0)}$ which is unique up to homotopy, such that the following diagram commutes up to homotopy.*



Remark 1.2.6. A map localizes the homotopy groups if and only if it localizes the homology groups. More precisely, suppose X and $X_{(0)}$ are connected CW complexes with $X_{(0)}$ $\mathbb{Q}$-local, and let $f : X \to X_{(0)}$ be a map which induces isomorphism on the fundamental group, then the following are equivalent [**GM**]:

(a) $f : X \to X_{(0)}$ is a localization defined as above.

(b) $f_* : \widetilde{H}_*(\widetilde{X};\mathbb{Q}) \to \widetilde{H}_*(\widetilde{X_{(0)}};\mathbb{Q})$ is a isomorphsim.

Definition 1.2.7. We say two CW complexes X and Y are *rational homotopy equivalent* if there exists a homotopy equivalence between their localizations:

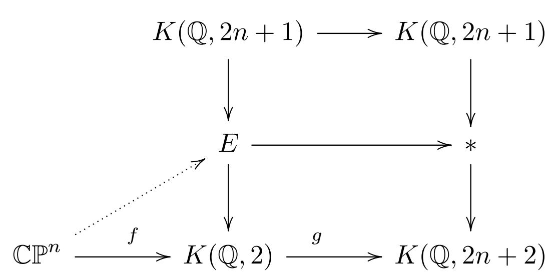
$$f : X_{(0)} \xrightarrow{\simeq} Y_{(0)}$$

1.2.1. Localization of simply-connected spaces. For simply-connected spaces, the localization can be constructed inductively from Postnikov towers.

Remark 1.2.8. Such construction can be extended to the more general so-called "nilpotent spaces" (CW complexes having nilpotent fundamental group which acts nilpotently on the higher homotopy groups) [**S2**]. But the resulting localization would also localize the fundamental group.

Example 1.2.9. Construction of $\mathbb{CP}^n_{(0)}$.

Let α be a generator of $H^2(\mathbb{CP}^n;\mathbb{Q})$. Let $f \in [\mathbb{CP}^n, K(\mathbb{Q},2)] \cong H^2(\mathbb{CP}^n;\mathbb{Q})$ be such that $f^*(\iota_2) = \alpha$, then f induces isomorphisms on $H^*(-;\mathbb{Q})$ through $* < 2n+2$. Let $g \in [K(\mathbb{Q},2), K(\mathbb{Q},2n+2)] = H^{2n+2}(K(\mathbb{Q},2);\mathbb{Q})$ be such that $g^*\iota_{2n+2} = \iota_2^{n+1}$. Pulling back the fibration $K(\mathbb{Q},2n+1) \to * \to K(\mathbb{Q},2n+2)$ via g, we get a fibration $K(\mathbb{Q},2n+1) \to E \to K(\mathbb{Q},2)$ with k-invariant ι_2^{n+1}.



The map g induces a morphism between the two spectral sequence of the two corresponding fibrations:

$H^p(K(\mathbb{Q}, 2n+2); H^q(K(\mathbb{Q}, 2n+1); \mathbb{Q})) \Rightarrow H^{p+q}(*; \mathbb{Q})$

$$\begin{array}{c|cccc} 2n+1 & \iota_{2n+1} & & & \\ & & & & \\ & & & & \iota_{2n+2} \\ \hline & & 2 & \cdots & 2n+2 \end{array}$$

$H^p(K(\mathbb{Q}, 2); H^q(K(\mathbb{Q}, 2n+1); \mathbb{Q})) \Rightarrow H^{p+q}(E; \mathbb{Q})$

$$\begin{array}{c|cccc} 2n+1 & \iota_{2n+1} & & & \\ & & & & \\ & & \iota_2 & \cdots & \iota_2^{n+1} \\ \hline & & 2 & \cdots & 2n+2 \end{array}$$

In the first spectral sequence, the class ι_{2n+1} is killed by a differential $d^k(\iota_2^{n+1}) = \iota_{2n+1}$, which implies that in the second spectral sequence, ι_{2n+1} is also killed at the stage. So we have:

$$\pi_*(E) = \begin{cases} \mathbb{Q} & * = 2 \text{ and } 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$H^*(E; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 2i \text{ and } i \leq n \\ 0 & \text{otherwise} \end{cases}$$

and by obstruction theory, the map $f : \mathbb{CP}^n \to K(\mathbb{Q}, 2)$ lifts to a map $\widehat{f} : \mathbb{CP}^n \to E$ which will be a localization map, so we constructed a $\mathbb{Q}$-local space $E = \mathbb{CP}^n_{(0)}$.

1.2.2. Localization of non-simply-connected spaces. For non-simply-connected spaces, the localization can be constructed from the fibrewise localization functor [**BK**].

Given a fibration of spaces $F \to E \to B$ with F simply-connected, the fibrewise localization functor gives a functorial commutative diagram:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \| \\ F_{(0)} & \longrightarrow & E' & \longrightarrow & B \end{array}$$

such that the bottom row is a fibration and the map $F \to F_{(0)}$ agrees with the localization of simply-connected spaces constructed from the Postnikov tower.

Given arbitrary CW complex X with fundamental group π, one can define $X_{(0)}$ to be $(E\pi \times_\pi \widetilde{X})'$, by applying the fibrewise localization functor to the fibration

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & E\pi \times_\pi \widetilde{X} \simeq X/\pi \simeq X \\ & & \downarrow \\ & & B\pi \end{array}$$

1.3. Homotopy cartesian and cocartesian square

The concept of homotopy cartesian and homotopy cocartesian square will be used later in the proof of the rational surgery existence theorem.

DEFINITION 1.3.1. A commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & D \end{array}$$

is called a *homotopy cartesian square* if the induced map from A to the homotopy pullback $P(k, g)$ is a weak homotopy equivalence. The homotopy pullback is defined to be $\{(b, \alpha, c) \in B \times D^I \times C \mid k(b) = \alpha(0), g(c) = \alpha(1)\}$

THEOREM 1.3.2. *Given a commutative square as above, the following are equivalent:*

(a) The square is a homotopy cartesian square.

(b) The induced map of the homotopy fibres $F_f \to F_g$ is a weak homotopy equivalence.

(c) The induced map of the homotopy fibres $F_h \to F_k$ is a weak homotopy equivalence.

DEFINITION 1.3.3. A commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & D \end{array}$$

is called a *homotopy cocartesian square* if the induced map from the homotopy pushout $M(f,h)$ to D is a weak homotopy equivalence. The homotopy pushout $M(f,h)$ is defined to be the double mapping cylinder $M_f \cup_A M_h = (B \cup (A \times I) \cup C)/\{(a,0) \sim f(a), (a,1) \sim h(a)\}$

THEOREM 1.3.4. *Given a commutative square as above, the following are equivalent:*

(a) The square is a homotopy cocartesian square.

(b) The induced map of the homotopy cofibres $C_f \to C_g$ is a weak homotopy equivalence.

(c) The induced map of the homotopy cofibres $C_h \to C_k$ is a weak homotopy equivalence.

1.4. Surgery theory

We will give a brief review of classical surgery theory in this section. The classical references are [**B**], [**W**] and [**R**].

DEFINITION 1.4.1. An *n-dimensional Poincaré complex* X is a finite CW complex with an orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$ and a *fundamental class* $[X] \in H_n(X; \mathbb{Z}^\omega)$ such that:

$$\cap[X] : H^*(X; \mathbb{Z}[\pi_1(X)]) \xrightarrow{\cong} H_{n-*}(X; \mathbb{Z}[\pi_1(X)]^\omega)$$

Given a Poincaré complex X, there are two obstructions to the existence of a closed manifold which is homotopy equivalent to X. The first obstruction is the existence of a degree 1 normal map which gives a candidate manifold to perform surgery on; the second obstruction is the vanishing of surgery obstruction, which can be computed algebraically, it determines if surgery can produce a manifold realizing the starting homotopy type.

1.4.1. Spivak normal fibration.

DEFINITION 1.4.2. A *Spivak normal fibration* of a n-dimensional Poincaré complex X is a $(k-1)$-spherical fibration $\nu_X : X \to BSG(k)$ together with a Spivak class $\alpha \in \pi_{n+k}(T\nu_X)$ such that the image of α under the composition of the Hurewicz map and the Thom isomorphism is the fundamental class of X. i.e.

$$\pi_{n+k}(T\nu_X) \to \widetilde{H}_{n+k}(T\nu_X, Z) \to H_n(X; \mathbb{Z})$$

$$\alpha \longmapsto [X]$$

THEOREM 1.4.3. [**B**, Spivak Uniqueness Theorem] *Any two Spivak normal fibrations* $(\nu, \alpha), (\nu', \alpha')$ *over* X *are related by a stable fibre homotopy equivalence* $b : \nu \overset{\simeq}{\to} \nu'$ *such that* $T(b)_*(\alpha) = \alpha'$. *Stabilizing* ν_X, *we call the resulting stable spherical fibration* the Spivak normal fibration of X.

REMARK 1.4.4. The Spivak normal fibration of a Poincaré complex is the homotopy theoretic analogue of the stable normal bundle of a manifold.

Given any n-dimensional Poincaré complex X, one can embed K in some sphere S^{n+k} for k large with regular neighborhood $(N^{n+k}, \partial N)$. Then the inclusion $\partial N \to N$ is homotopy equivalent to a fibration with fibre a homotopy S^{k-1} [**B**, Theorem I.4.4]. Pull back this spherical fibration through the homotopy equivalence $X \to N$, the resulting spherical fibration $E \to X$ is the desired Spivak normal fibration of X.

1.4.2. Degree one normal map. The first obstruction to the surgery problem vanishes if there exists a bundle reduction of the Spivak normal fibration of X, i.e., one asks: does there exist a vector bundle $\xi : X \to BSO$ such that:

$$\begin{array}{ccc} & & BSO \\ & \overset{\xi}{\nearrow} & \downarrow \\ X & \xrightarrow{\nu_X} & BSG \end{array}$$

commutes up to homotopy. If so, then there exists a degree 1 normal map.

DEFINITION 1.4.5. A *degree 1 normal map* $(f, b) : (M, \nu_M) \to (X, \xi)$ is a map $f : M \to X$ from a n-dimensional manifold M to a n-dimensional Poincaré complex X such that $f_*[M] = [X]$, together with a bundle map $b : \nu_M \to \xi$ covering f where ν_M is the stable normal bundle of M.

DEFINITION 1.4.6. A *normal bordism* is a degree 1 normal map

$$((F, B); (f, b), (f', b')) : (W; M, M') \to X \times (I; 0, 1)$$

where $(W; M, M')$ is a cobordism. We say that the normal maps (f, b) and (f', b') are *normally bordant* to each other.

The set of equivalence classes (up to normal bordism) of degree 1 normal map with range X is called the the *normal structure set* of X, and is denoted by $\mathcal{N}(X)$.

REMARK 1.4.7. [**B**][**R**][**MM**] If M is a smooth manifold, the normal structure set $\mathcal{N}_O(M)$ is isomorphic to $[M, G/O]$ and if M is a PL manifold, the normal structure set $\mathcal{N}_{PL}(M)$ is isomorphic to $[M, G/PL]$.

THEOREM 1.4.8. [**B**][**R**][**MM**] *Let X^n be a Poincaré complex and ξ^k a vector bundle over X^n. There exists a degree 1 normal map $(f, b) : (M^n, \nu_M) \to (X^n, \xi^k)$ if and only if ξ^k is a reduction of the Spivak normal fibration ν^k_X.*

PROOF. ($\Longleftarrow$) If ξ^k is a reduction of the Spivak normal fibration ν^k_X, the proper fibre homotopy equivanece $\xi^k \to \nu^k_X$ induces a homotopy equivalence $T\xi^k \to T\nu^k_X$. The Spivak class $\alpha \in \pi_{n+k}(T\nu^k_M)$ corresponds to a homotopy class $c \in \pi_{n+k}(T\xi^k)$ such that, under the Hurewicz map and the Thom isomorphism, $h(c) \cap U = [X]$. One can perform the Thom-Pontryagin construction. Let $g : S^{n+k} \to T\xi^k$ represent the class c. By the Thom transversality theorem, we can assume g is transverse to the zero section X in $T(\xi^k)$. Then $M = g^{-1}(X)$ is a n-dimensional smooth manifold whose normal bundle ν^k_M is mapped to ξ^k by g. By construction, the restricted map $g|_M : M^n \to X^n$ is a degree 1 map.

($\Longrightarrow$) If there exist a degree 1 normal map $(f, b) : (M^n, \nu_M) \to (X^n, \xi^k)$, the sphere bundle $S\xi^k$ is a Spivak normal fibration with Spivak class $T(b)c_M$ where $c_M \in \pi_{n+k}(T\nu_M)$ is the class representing the collapsing map $c_M : S^{n+k} \to S^{n+k}/(S^{n+k}\ D(\nu_M)) = T\nu_M$. Then by the Spivak uniqueness theorem, ξ^k is a reduction of ν^k_X.

1.4.3. Surgery obstruction. The surgery obstruction lives in the L-group of the group ring $\mathbb{Z}[\pi_1(X)]$, it is the kernel form of the candidate degree 1 normal map. The L-group can be computed algebraically.

THEOREM 1.4.9. *Given a degree 1 normal map $(f, b) : (M^m, \nu_M) \to (X, \xi)$ $(m \geq 5)$, there is a surgery obstruction*

$$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

such that $\sigma_*(f,b)=0$ if and only if (f,b) is normally bordant to a homotopy equivalence.

EXAMPLE 1.4.10. [**R**] In the case that X is simply-connected, we have

$$L_m(\mathbb{Z}) = \begin{cases} \mathbb{Z} & m \equiv 0 \pmod 4 \\ \mathbb{Z}_2 & m \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

REMARK 1.4.11. In the $4k$ dimensional simply-connected case, the surgery obstruction vanishes if and only if the symmetric forms $(H^{2k}(M;\mathbb{Z}),\lambda)$ and $(H^{2k}(X;\mathbb{Z}),\lambda)$ are stably isomorphic, i.e.

$$(H^{2k}(M;\mathbb{Z}),\lambda) \oplus H(\mathbb{Z}^m) \cong (H^{2k}(X;\mathbb{Z}),\lambda) \oplus H(\mathbb{Z}^\ell)$$

where $H(\mathbb{Z}^m)$ and $H(\mathbb{Z}^\ell)$ are hyperbolic forms. And this happens if and only if the two forms have the same signature

$$\sigma(H^{2k}(M;\mathbb{Z}),\lambda) = \sigma(H^{2k}(X;\mathbb{Z}),\lambda)$$

1.5. Pontryagin classes

Pontryagin classes are characteristic classes that play an important role in surgery theory. The Pontryagin numbers are homeomorphism invariants of smooth manifold.

THEOREM 1.5.1. [**MS**] *Let $p_i(\gamma) \in H^{4i}(BSO;\mathbb{Z})$ be the Pontryagin class of the universal bundle γ over the smooth orientable classifying space BSO. The rational cohomology ring of BSO is generated by these Pontryagin classes:*

$$H^*(BSO;\mathbb{Q}) = \mathbb{Q}[p_1(\gamma), p_2(\gamma), \cdots]$$

and there are no polynomial relations among the $p_i(\gamma)$'s.

DEFINITION 1.5.2. Let M^{4n} be a smooth, compact, oriented manifold. For each partition $I = i_1, \dots, i_r$ of n, the *Ith Pontryagin number* $p_I[M] = p_{i_1}\cdots p_{i_r}[M]$ is defined to be the integer

$$\langle p_{i_1}(\tau_M)\cdots p_{i_r}(\tau_M), [M]\rangle$$

where $p_{i_k}(\tau_M) \in H^{4i_k}(M;\mathbb{Z})$ is the i_kth Pontraygin class of the tangent bundle of M.

REMARK 1.5.3. In the 1960s, Novikov proved the famous result that rational Pontryagin classes of the tangent bundle are homeomorphism invariants for manifolds.

DEFINITION 1.5.4. Given a formal power series $f(t)$, the associated *multiplicative sequence* is a sequence of polynomials in the variables x_i

$$K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \cdots$$

such that

$$K_k(\sigma_1, \cdots, \sigma_k) = \text{the degree } k \text{ homogeneous part of } \; f(t_1)\cdots f(t_k)$$

where σ_i is the ith elementary symmetric polynomial on the variables $t_1, t_2, \cdots, t_k$, i.e.

$$1 + \sigma_1 + \sigma_2 + \cdots + \sigma_k = (1 + t_1)\cdots(1 + t_k)$$

EXAMPLE 1.5.5. The L polynomial $L_k(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the degree k polynomial of the multiplicative sequence associated to the power series

$$f(t) = \sqrt{t}/\tanh\sqrt{t} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots$$

.

For example, when $k = 2$, if we formally write $p_1 = \sigma_1(t_1, t_2)$, $p_2 = \sigma_2(t_1, t_2)$, then

$$\begin{aligned}
L_2(p_1, p_2) &= \text{the degree 2 homogeneous part of } \left(\frac{\sqrt{t_1}}{\tanh\sqrt{t_1}}\right) \times \left(\frac{\sqrt{t_2}}{\tanh\sqrt{t_2}}\right) \\
&= \text{the degree 2 homogeneous part of } (1 + \frac{1}{3}t_1 - \frac{1}{45}t_1^2 + \cdots) \times (1 + \frac{1}{3}t_2 - \frac{1}{45}t_2^2 + \cdots) \\
&= \frac{1}{9}t_1 t_2 - \frac{1}{45}(t_1^2 + t_2^2) \\
&= \frac{1}{9}t_1 t_2 - \frac{1}{45}((t_1 + t_2)^2 - 2t_1 t_2) \\
&= -\frac{1}{45}(t_1 + t_2)^2 + \frac{7}{45}t_1 t_2 \\
&= -\frac{1}{45}p_1^2 + \frac{7}{45}p_2
\end{aligned}$$

THEOREM 1.5.6. (Hirzebruch signature theorem) *For any smooth closed oriented manifold M^{4k} with Pontryagin classes $p_i = p_i(\tau_M)$, fundamental class $[M^{4k}]$, and signature $\sigma(M^{4k})$,*

$$\sigma(M^{4k}) = \langle L_k(p_1, \cdots, p_k), [M^{4k}]\rangle$$

REMARK 1.5.7. The signature theorem indicats that the coefficients in the L polynomial can be also obtained in the following way. The signature function $M \mapsto \sigma(M)$ gives rise to an algebra homomorphism from the rational cobordism algebra $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$. And since the algebra

$$\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$$

and $\sigma(\mathbb{CP}^{2k}) = 1$, we can solve for the coefficient of the L polynomial. For example, $L_3(p_1, p_2, p_3) = ap_3 + bp_1p_2 + cp_1^3$. By the signature theorem, we have

$$\begin{cases} 1 = ap_3[\mathbb{CP}^6] + bp_1p_2[\mathbb{CP}^6] + cp_1^3[\mathbb{CP}^6] \\ 1 = ap_3[\mathbb{CP}^2 \times \mathbb{CP}^4] + bp_1p_2[\mathbb{CP}^2 \times \mathbb{CP}^4] + cp_1^3[\mathbb{CP}^2 \times \mathbb{CP}^4] \\ 1 = ap_3[\mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2] + bp_1p_2[\mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2] + cp_1^3[\mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2] \end{cases}$$

The Pontryagin numbers of $\mathbb{CP}^{2k}$ can be computed by the following formula [**MS**]

$$p_{i_1} \cdots p_{i_r}[\mathbb{CP}^{2k}] = \binom{2k+1}{i_1} \cdots \binom{2k+1}{i_r}$$

so the above system of linear equations can be rewritten as

$$\begin{cases} 1 = 35a + 147b + 343c \\ 1 = 30a + 105b + 225c \\ 1 = 27a + 81b + 162c \end{cases}$$

Hence

$$\begin{cases} a = \frac{62}{945} \\ b = -\frac{13}{945} \\ c = \frac{2}{945} \end{cases}$$

So we have computed that

$$L_3(p_1, p_2, p_3) = \frac{62}{945}p_3 + -\frac{13}{945}p_1p_2 + \frac{2}{945}p_1^3$$

REMARK 1.5.8. [**MS**, Problem 19-C] Let B_n be the n-th Bernoulli number. The coefficient of p_n in the L-polynomial $L_n(p_1, \ldots, p_n)$ is equal to $2^{2n}(2^{2n-1} - 1)B_n/(2n)!$.

CHAPTER 2

Rational Surgery Preliminaries

2.1. $\mathbb{Q}$-Poincaré complex

In rational surgery theory, the starting space carrying all the desired rational homotopy data is a $\mathbb{Q}$-local space satisfying the Poincaré duality in rational coefficients.

DEFINITION 2.1.1. An *n-dimensional* $\mathbb{Q}$*-Poincaré complex* is a CW complex X that is rational homotopy equivalent to a finite CW complex, with an orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$ and a *fundamental class* $[X] \in H_n(X; \mathbb{Q}^\omega)$ such that:

$$\cap[X] : H^*(X; \mathbb{Q}[\pi_1(X)]) \xrightarrow{\cong} H_{n-*}(X; \mathbb{Q}[\pi_1(X)]^\omega)$$

To determine if a CW complex X is a $\mathbb{Q}$-Poincaré complex, we need to first check if X has the rational homotopy type of a finite complex. The following lemma gives necessary and sufficient condition in the case when X has finite fundamental group.

LEMMA 2.1.2. [**DM**, Theorem 3.5] *Given a CW complex X with finite fundamental group G and universal cover $\widetilde{X}$, the following are equivalent:*

(a) *X is rational homotopy equivalent to a finite complex.*

(b) *$H_*(\widetilde{X}; \mathbb{Q})$ is finitely generated, and for all $g \in G - \{e\}$, the Lefschetz number*

$$\sum (-1)^i \operatorname{tr}(g_* : H_i(\widetilde{X}; \mathbb{Q}) \to H_i(\widetilde{X}; \mathbb{Q})) = 0$$

(c) *$H_*(\widetilde{X}; \mathbb{Q})$ is finitely generated and $\sum_{i=0}^{\infty} (-1)^i [H_i(\widetilde{X}; \mathbb{Q})] = 0 \in \widetilde{K}_0(\mathbb{Q}G)$.*

REMARK 2.1.3. If X is simply-connected with $\sum \dim H_i(X; \mathbb{Q}) < \infty$, there always exists a finite CW complex that is rational homotopy equivalent to X.

2.2. Rational Spivak normal fibration

Given any n-dimensional $\mathbb{Q}$-Poincaré complex X, there exists a finite CW complex K and a homotopy equivalence $f : X_{(0)} \to K_{(0)}$. One can embed K in some sphere S^{n+k} for k large with regular neighborhood $(N^{n+k}, \partial N)$. Since K is also a $\mathbb{Q}$-Poincaré complex, the inclusion $\partial N \to N$ is homotopy equivalent to a fibration $E \to N$ whose fibre F is rational homotopy equivalent to S^{k-1}. Localizing the fibration we get another fibration $S^{k-1}_{(0)} \to E_{(0)} \to N_{(0)} \simeq K_{(0)}$. Now we pull back this fibration through the map $X \to X_{(0)} \to N_{(0)}$ to get a fibration $S^{k-1}_{(0)} \to E' \to X$. We call this rational spherical fibration $\nu_X : X \to BSG(k)_{(0)}$ the rational Spivak normal fibration of X.

Moreover, if X is $\mathbb{Q}$-local, there exists a Spivak class $\alpha \in \pi_{n+k}(T\nu_X)$ such that the image of α under the composition of the Hurewicz map and the Thom isomorphism is the fundamental class of X. i.e.

$$\pi_{n+k}(T\nu_X) \to \widetilde{H}_{n+k}(T\nu_X; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \cong H_n(X; \mathbb{Q}) \cong \mathbb{Q}$$

$$\alpha \longmapsto [X]$$

Remark 2.2.1. Since $BSG_{(0)}$ is contractible, the rational Spivak normal fibration $\nu_X : X \to BSG_{(0)}$ is fibre homotopy equivalent to any other rational spherical fibration $\nu' : X \to BSG_{(0)}$. Moreover, any map $\xi : X \to BSO_{(0)}$ is a reduction of ν_X, i.e. the diagram

$$\begin{array}{ccc} & & BSO_{(0)} \\ & \overset{\xi}{\nearrow} & \downarrow \\ X & \xrightarrow{\nu_X} & BSG_{(0)} \simeq * \end{array}$$

commutes up to homotopy.

2.3. Rational degree 1 normal map

Definition 2.3.1. Let X be a n-dimensional $\mathbb{Q}$-Poincaré complex with a specified fundamental class $[X] \in H_n(X; \mathbb{Q})$ and a vector bundle ξ over X. A *rational degree 1 normal map* $(f, b) : (M, \nu_M) \to (X, \xi)$ is a map $f : M \to X$ from a n-dimensional closed oriented manifold M to X together with a bundle map $b : \nu_M \to \xi$ covering f, such that $r_* f_* [M] = [X]$ with $r_* : H_n(X; \mathbb{Z}) \to H_n(X; \mathbb{Q})$ and ν_M is the normal bundle of M.

DEFINITION 2.3.2. If we do not specify a choice of fundamental class $[X] \in H_n(X;\mathbb{Q})$, a *rational nonzero degree normal map* is a map $f : M \to X$ covered by a bundle map $b : \nu_M \to \xi$ such that $f_*[M] \neq 0 \in H_n(X;\mathbb{Q})$.

REMARK 2.3.3. Let X^n be a rational Poincaré complex and ξ^k a vector bundle over X^n. There exists a rational degree 1 normal map $(f,b) : (M^n, \nu_M) \to (X^n, \xi^k)$ if and only if there exists a homotopy class $c \in \pi_{n+k}(T\xi^k)$ such that, under the Hurewicz map, the Thom isomorphism and the coefficient map $r_* : H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Q})$, $r_*(h(c) \cap U) = [X]$. If there is such a homotopy class c, we can perform the Thom-Pontryagin construction to get the rational degree 1 normal map as described in the ordinary case.

2.4. Rational surgery obstruction

THEOREM 2.4.1. [**A**][**TW**] *Given a rational degree 1 normal map* $(f,b) : (M^m, \nu_M) \to (X, \xi)$ $(m \geq 5)$*, there is a surgery obstruction*

$$\sigma_*(f,b) \in L_m(\mathbb{Q}[\pi_1(X)])$$

such that $\sigma_*(f,b) = 0$ *if and only if* (f,b) *is normally bordant to a rational homotopy equivalence.*

EXAMPLE 2.4.2. [**R**] In the case that X is simply-connected,

$$L_m(\mathbb{Q}) = \begin{cases} \mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}_2 \oplus \bigoplus_\infty \mathbb{Z}_4 & m \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

2.5. Homotopy cartesian and cocartesian squares involving localization maps

In the proof of the rational surgery existence theorem, we will use the following lemma.

LEMMA 2.5.1. [**TW**, Lemma 6.1] *Consider the square of connected CW complexes*

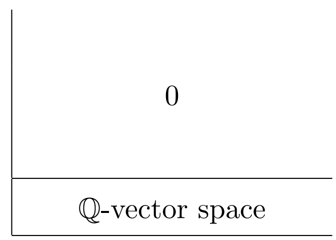
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ A_{(0)} & \xrightarrow{g} & B_{(0)} \end{array}$$

Suppose that g induces an isomorphism on π_1 and the vertical maps are localizations. Then if the square is a homotopy cartesian square, it is a homotopy cocartesian square. If we further suppose $\pi_1 A = 1$, the converse holds.

Proof. ($\Longrightarrow$): Suppose the square above is a homotopy cartesian square. Let F denote the homotopy fibre of h and k. There is a homology spectral sequence with

$$H_p(B_{(0)}, A_{(0)}; H_q(F)) \Rightarrow H_{p+q}(B, A)$$

It is clear that $\widetilde{H}_*(F)$ are torsion and $H_*(B_{(0)}, A_{(0)})$ are $\mathbb{Q}$-vector spaces, so the E_2-stage is like:



All the differentials vanish. We have $E_2 \cong E_\infty$, and so $H_*(B, A) \xrightarrow{\cong} H_*(B_{(0)}, A_{(0)})$. By the Whitehead Theorem, the homotopy cofibres C_f and C_g are weak homotopy equivalent, which implies that the square is a homotopy cocartesian square.

($\Longleftarrow$): Suppose the square is a homotopy cocartesian square and $\pi_1 A = 0$. Then the mapping cones C_h and C_k are homotopy equivalent. For the fibration sequence $F_A \to A \to A_{(0)}$, it is clear that $\widetilde{H}_p(A_{(0)}; \widetilde{H}_q(F_A)) = 0$ for all p, q. And since $\pi_1 A = 0$, we have the Serre exact sequence on homology, so $F_A \to A \to A_{(0)}$ is also a cofibration sequence. Then we have the Puppe sequence:

$$F_A \to A \to A_{(0)} \to \Sigma F_A \to \Sigma A \to \Sigma A_{(0)} \to \cdots$$

where every 2 consecutive maps in the sequence form a cofibration sequence up to homotopy. So $A \to A_{(0)} \to \Sigma F_A$ is also a cofibration sequence. Then we have $\Sigma F_A \simeq C_h \simeq_{weak} C_k \simeq \Sigma F_B$. Then by the fact that $\widetilde{H}_{*+1}(\Sigma X) \cong \widetilde{H}_*(X)$, we have $\widetilde{H}_*(F_A) \cong \widetilde{H}_*(F_B)$, and by the Whitehead theorem, F_A and F_B are weak homotopy equivalent. So the square is a homotopy cartesian square. $\square$

CHAPTER 3

Simply-connected rational surgery

3.1. Sullivan's Theorem

In [**S1**, Theorem 13.2], Sullivan stated a theorem on realizing a given rational homotopy type by constructing a closed manifold using surgery theory, followed by a sketch of the proof. Here we restate and prove it carefully.

THEOREM 3.1.1. *Let X be an $n = 4k$ dimensional simply-connected, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex. There exists a simply-connected smooth closed $4k$ dimensional manifold M, and a $\mathbb{Q}$-homotopy equivalence $f : M \longrightarrow X$ if and only if:*

***Case 1**: signature $\sigma(X) = 0$*

There exists cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \le i \le k$ and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q}) \cong \mathbb{Q}$ such that:

(i) $L_k(p_1, \ldots, p_k) = 0 \in H^{4k}(X;\mathbb{Q})$

(ii) *The symmetric bilinear form $H^{2k}(X;\mathbb{Q}) \times H^{2k}(X;\mathbb{Q}) \to \mathbb{Q}$ defined as $\langle \cdot \cup \cdot, \mu\rangle$ is isomorphic to $m\langle 1\rangle \oplus m\langle -1\rangle$.*

***Case 2**: signature $\sigma(X) \neq 0$*

There exists cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \le i \le k$, and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q}) \cong \mathbb{Q}$ such that

(i) $\langle L_k(p_1, \ldots, p_k), \mu\rangle = \sigma(X)$

(ii) *The symmetric bilinear form $H^{2k}(X;\mathbb{Q}) \times H^{2k}(X;\mathbb{Q}) \to \mathbb{Q}$ defined as $\langle \cdot \cup \cdot, \mu\rangle$ is isomorphic to $m\langle 1\rangle \oplus n\langle -1\rangle$*

(iii) *There exists a closed smooth $4k$ dimensional manifold N such that*

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle$$

for all partitions I of k.

If the choice of cohomology classes p_i and fundamental class μ satisfy all the above conditions ((i) and (ii) in case 1; (i),(ii) and (iii) in case 2), surgery theory will construct a $\mathbb{Q}$-homotopy equivalence $f : M \to X$ which satisfies $f^*p_i = p_i(\tau_M)$, where $p_i(\tau_M)$ are the Pontryagin classes of the tangent bundle of M, and in case 2, the Pontryagin numbers $p_I[M] = \langle p_I, \mu\rangle$ for all partitions I of k.

Proof. ($\Longrightarrow$): Suppose there exists a simply-connected closed $4k$ dimensional manifold M, and a $\mathbb{Q}$-homotopy equivalence $f : M \longrightarrow X$. Then the induced map on the cohomology rings $f^* : H^*(X;\mathbb{Q}) \xrightarrow{\cong} H^*(M;\mathbb{Q})$ is a isomorphism. Let $p_i \in H^{4i}(X;\mathbb{Q})$ be the cohomology classes such that $f^*p_i = p_i(\tau_M)$, $1 \leq i \leq k$, then:

$$\begin{aligned} \langle L_k(p_1,\ldots,p_k), f_*[M]\rangle &= \langle L_k(f^*p_1,\ldots,f^*p_k),[M]\rangle \\ &= \langle L_k(p_1(\tau_M),\ldots,p_k(\tau_M)),[M]\rangle \\ &= \sigma(M) \\ &= \sigma(X) \end{aligned}$$

For the case $\sigma(X) \neq 0$, let $\mu = f_*[M]$.

For (ii) in both cases, the intersection form on $H^{2k}(X;\mathbb{Q})$ with respect to the fundamental class $f_*[M]$ is isomorphic to the rational intersection form of M, which is the image of a nonsingular symmetric form over $\mathbb{Z}$. By the Witt Cancelation Theorem and the fact that the image of the Witt ring $W(\mathbb{Z})$ in $W(\mathbb{Q})$ consists exactly of classes of the form $m\langle 1\rangle \oplus n\langle -1\rangle$ over $\mathbb{Q}$, (ii) is satisfied.

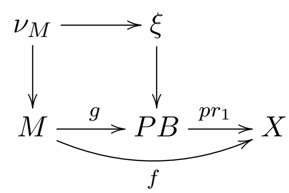
For (iii) in Case 2 , just let $N = M$.

($\Longleftarrow$): We will prove that under the hypothesis, there exists a rational degree 1 normal map such that the surgery obstruction vanishes. The proof will be carried out in §3.2 and §3.3.

3.2. Rational degree 1 normal maps

We will first prove that given any set of cohomology classes $p_i \in H^{4i}(X;\mathbb{Q})$ for $1 \leq i \leq k$, one can construct a nonzero degree rational normal map pulling back the p_i's to the Pontryagin classes of the manifold. And this is sufficient to give the desired normal map in the case that $\sigma(X) = 0$. For the case when $\sigma(X) \neq 0$, we prove that by adding the hypothesis condition (iii), one can construct a rational degree 1 normal map that maps the fundamental class of the candidate manifold to a specific fundamental class of the local space that will also work in the surgery obstruction step.

LEMMA 3.2.1. *Let X be an $n = 4k$ dimensional simply-connected, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex, equipped with cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \leq i \leq k$. There always exists a $\mathbb{Q}$-Poincaré complex PB, a localization $pr_1 : PB \to X$, and a rational nonzero degree normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$ where $f^*p_i = p_i(\tau_M)$ for $f = pr_1 \circ g$.*



PROOF. The homotopy classes of maps from a space to the Eilenberg-Maclane space $K(\mathbb{Q},4i)$ are in $1-1$ correspondence with the $4i$ dimensional rational cohomology classes of the space, so the p_i's define a map $p : X \xrightarrow{(p_1,\ldots,p_k)} \Pi K(\mathbb{Q},4i) \simeq BSO_{(0)}$. For $m >> n$, define $\overline{p} : BSO(m) \xrightarrow{(\overline{p_1},\ldots,\overline{p_i},\ldots)} \prod K(\mathbb{Q},4i)$, where $\overline{p_i} \in H^{4i}(BSO(m);\mathbb{Q})$ are such that

$$(1+\overline{p_1}+\ldots+\overline{p_i}+\ldots)(1+p_1(\gamma^m)+\ldots+p_i(\gamma^m)+\ldots) = 1$$

where γ^m is the universal plane bundle over $BSO(m)$.

Let PB be the homotopy pull-back space of p and $\overline{p}$ just defined. Let ξ^m denote the pullback bundle of γ^m over PB. Then by Theorem 18.3 in [**MS**] asserting that the Hurewicz homomorphism of an $(m-1)$-connected space is a $\mathcal{C}$-isomorphism up to dimension $2m-1$, we have

$$\begin{aligned} \pi_{n+m}(T\xi^m)\otimes\mathbb{Q} &\cong H_{n+m}(T\xi^m)\otimes\mathbb{Q} \\ &\cong H_n(PB;\mathbb{Q}) \quad \text{(Thom Isomorphism)} \\ &\cong H_n(X;\mathbb{Q}) \\ &\cong \mathbb{Q} \end{aligned}$$

So one can choose a class $\alpha \in \pi_{n+m}(T\xi^m)$ such that the image of α under the above isomorphisms is nonzero, and perform the Thom-Pontryagin construction. The class α is represented by a map $g : S^{m+n} \longrightarrow T\xi^m$ of nonzero degree. Deform g so that it is transverse to the zero-section PB, then define $M = g^{-1}(PB)$, where $g : M \to PB$ is now covered by the bundle map $\hat{g} : \nu_M = g^*\xi \to \xi$.

$$\begin{array}{ccccc} \nu_M & \longrightarrow & \xi & \longrightarrow & \gamma^m \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{g} & PB & \xrightarrow{pr_2} & BSO(m) \\ & \searrow^{f} & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \bar{p}} \\ & & X & \xrightarrow{p} & \Pi K(\mathbb{Q},4i) \end{array}$$

□

We now generalize Lemma 3.2.1, strengthening both the hypothesis and the conclusion to construct a rational degree 1 normal map such that the fundamental class of M maps to a specified fundamental class $\mu \in H_n(X;\mathbb{Q})$.

LEMMA 3.2.2. *Let X be an $n = 4k$ dimensional simply-connected, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex, together with cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \le i \le k$, and a fixed fundamental class $\mu \in H_n(X;\mathbb{Z}) \cong \mathbb{Q}$. Suppose there exists a closed $4k$ dimensional manifold N such that*

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle$$

for all partitions I of k, then there exists a rational degree 1 normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$ where $pr_1 : PB \to X$ is a localization and for $f = pr_1 \circ g : M \to X$, we have $f_*[M] = \mu$ and $f^*(p_i) = p_i(\tau_M)$.

Proof. With the additional hypothesis condition (iii) in Theorem 3.1.1, we will prove that there exists a "correct" class $\alpha \in \pi_{n+m}(T\xi^m)$ such that if we perform Thom-Pontryagin construction using α, the corresponding normal map would satisfy $f_*[M] = \mu$. We will first construct a three-level commutative diagram.

Condition (iii) says, there exists a closed manifold N such that for all partitions I of k

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle = \langle \overline{p}_I, \overline{p}_*^{-1}(p_*\mu)\rangle$$

which implies $\langle p_I(\nu_N), [N]\rangle = \langle p_I(\gamma^m), \overline{p}_*^{-1}(p_*\mu)\rangle$. Since $H^*(BSO; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots]$ and $H_n(BSO(m); \mathbb{Q}) \cong \mathrm{Hom}(H^n(BSO(m); \mathbb{Q}), \mathbb{Q})$, these congruences imply that $\overline{p}_*^{-1}(p_*\mu)$ lies in the image of the manifold N under the homomorphism $\nu : \Omega_n^{SO} \to H_n(BSO; \mathbb{Q})$ defined by $\nu(M) = \nu_{M*}[M]$ where ν_M is the classifying map for the normal bundle of M:

$$\begin{array}{ccc} \nu_M & \longrightarrow & \gamma \\ \downarrow & & \downarrow \\ M & \xrightarrow{\nu_M} & BSO \end{array}$$

Note that ν can also be interpreted as the map ν in the following diagram:

$$\begin{array}{ccc} \Omega_n^{SO} \cong \lim_{m\to\infty} \pi_{n+m}(T\gamma^m) & \xrightarrow{h} & \lim_{m\to\infty} H_{n+m}(T\gamma^m; \mathbb{Z}) \\ \Big\downarrow \nu & & \Big\downarrow \cap U \\ H_n(BSO; \mathbb{Q}) & \xleftarrow[i_*]{} & H_n(BSO; \mathbb{Z}) \end{array}$$

Then since $\overline{p}_*^{-1}(p_*\mu) \in \mathrm{Im}\ \nu$, there exists a class $\beta \in \pi_{n+m}(T\gamma^m)$ such that $\nu(\beta) = i_*(h(\beta) \cap U) = \overline{p}_*^{-1}(p_*\mu) \in H_n(BSO; \mathbb{Q})$.

Let $S^{m-1}_{(0)} \to \nu_X \to X$ be the rational Spivak normal fibration of the $\mathbb{Q}$-Poincaré complex X. Let $\widetilde{\nu}_X = p^*(S\gamma^m_{(0)})$, where $S^{m-1}_{(0)} \to S\gamma^m_{(0)} \to BSO(m)_{(0)}$ is the localization of the fibre bundle $S^{m-1} \to S\gamma^m \to BSO(m)$. As mentioned in Remark 2.1, since $BSG_{(0)}$ is contractible, ν_X is fibre homotopy equivalent to $\widetilde{\nu}_X$. So there exists a class $c_X \in \pi_{m+n}(T\widetilde{\nu}_X)$, such that under the Hurewicz and Thom map, c_X is mapped to $\mu \in H_n(X; \mathbb{Z}) \cong \mathbb{Q}$.

Then we have the following main diagram, which include three squares in the level of base spaces, spherical fibrations and Thom spaces respectively. It is easy to check that each vertical map is a localization map.

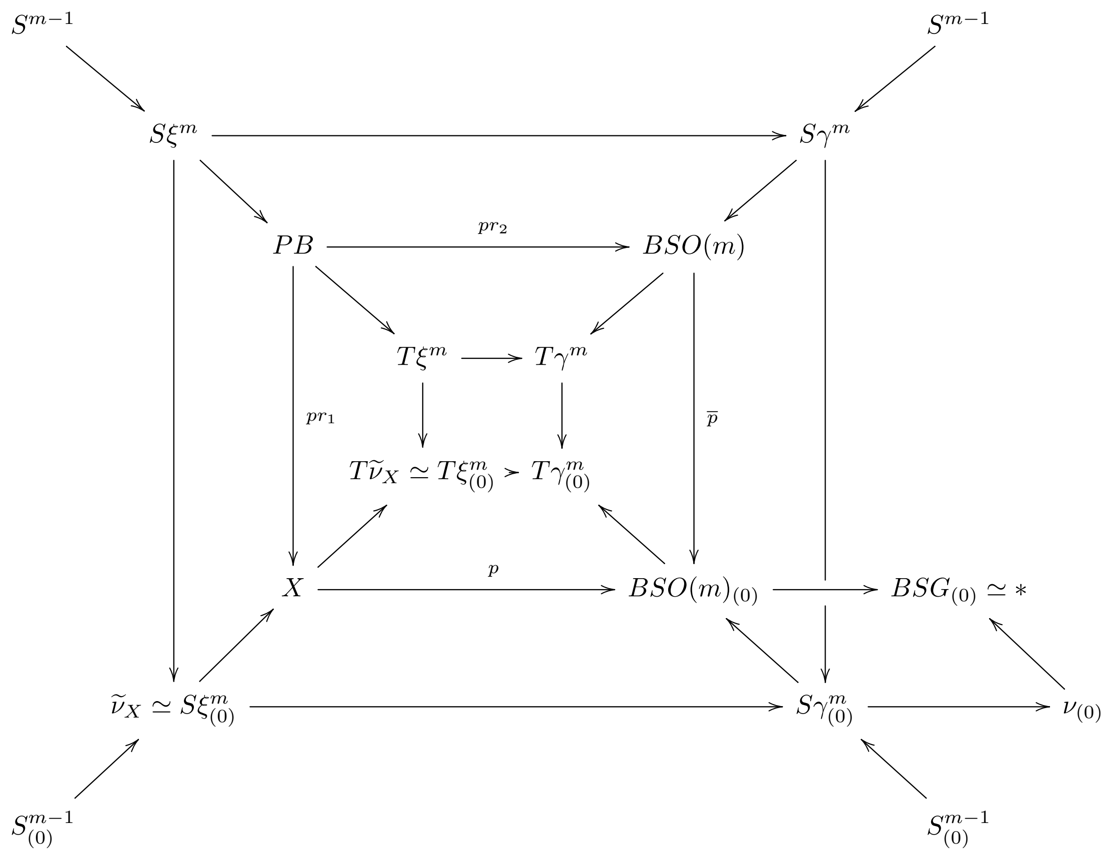


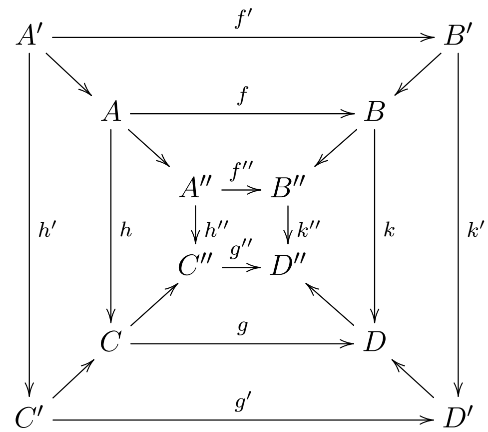
Figure 1. Main Diagram

To prove the existence of a "correct" class $\alpha \in \pi_{n+m}(T\xi^m)$ that gives the desired rational degree 1 normal map in Lemma 3.2.2, we will prove the following property of the diagram:

Lemma 3.2.3. *The inner-most square of Thom spaces in the above diagram is a homotopy cartesian square.*

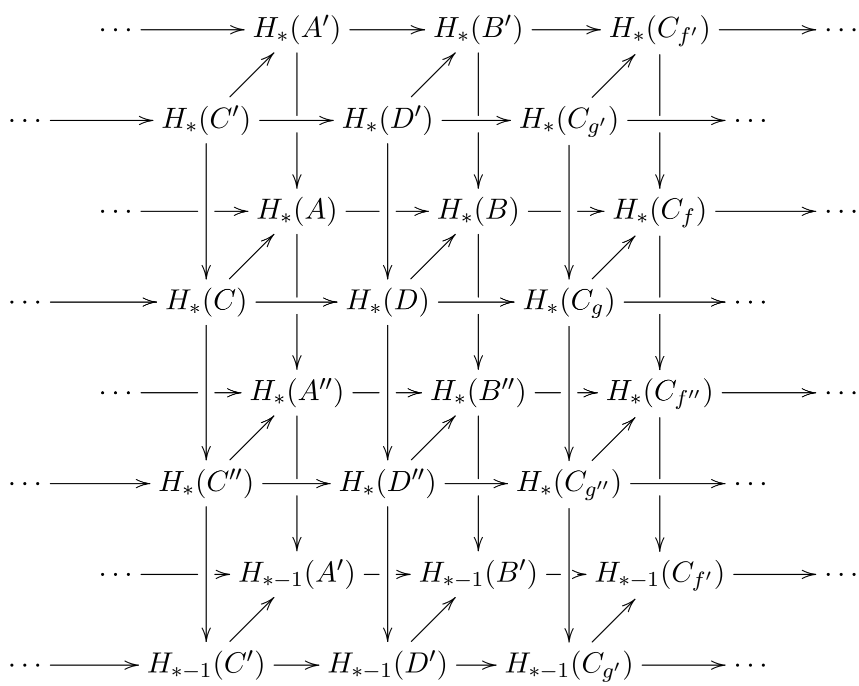
Proof. To prove this lemma, we will use Lemma 2.5.1 in both directions. We will also need the following easy lemma.

LEMMA 3.2.4. *Consider the diagram of CW complexes:*



Suppose that the outer-most square and middle square are homotopy cocartesian squares, and the four diagonals are cofibration sequences. Then the inner-most square is a homotopy cocartesian square.

PROOF. The proof follows from the following diagram of homology exact sequences of cofibrations. By the homotopy cocartesian square assumption, we have $H_*C_f \cong H_*C_g$ and $H_*C_{f'} \cong H_*C_{g'}$, then the five lemma implies $H_*C_{f''} \cong H_*C_{g''}$. Thus the inner-most square is a homotopy cocartesian square.



□

Now we get back to the proof of Lemma 3.2.3 asserting that the inner-most square of Thom spaces in the main diagram is a homotopy cartesian square. The middle square of base space and the outer-most square of spherical fibrations in the main diagram are homotopy cartesian squares by construction. By Lemma 2.5.1, they are homotopy cocartesian squares. Notice that for a spherical bundle ν over a space X, $\nu \to X \to T\nu$ is a cofibration sequence, so by Lemma 3.2.4, the inner-most square of Thom space is a homotopy cocartesian square. Since the Thom spaces are simply-connected, we can then apply the other direction of 2.5.1, so this square of Thom space is a homotopy cartesian square. This completes the proof of Lemma 3.2.3. □

Back to the proof of Lemma 3.2.2 asserting the existence of the desired rational degree 1 normal map. In the following diagram, the homotopy group of the Thom space maps to the homology of the base space through Hurewicz and Thom maps:

$$\begin{array}{ccccc}
\pi_{n+m}(T\xi^m) & & \xrightarrow{T_{pr_2 *}} & & \pi_{n+m}(T\gamma^m) \\
& \searrow & & \swarrow & \\
& H_n(PB) & \xrightarrow{pr_{2*}} & H_n(BSO(m)) & \\
{\scriptstyle Tpr_{1*}}\downarrow & \downarrow{\scriptstyle pr_{1*}} & & \downarrow{\scriptstyle \overline{p}_*} & \downarrow{\scriptstyle T\overline{p}_*} \\
& H_n(X) & \xrightarrow{p_*} & H_n(BSO(m)_{(0)}) & \\
& \nearrow & & \nwarrow & \\
\pi_{n+m}(T\widetilde{\nu}_X) & & \xrightarrow{T_{p*}} & & \pi_{n+m}(T\gamma^m_{(0)})
\end{array}$$

We now claim that $\beta \in \pi_{n+m}(T\gamma^m)$ and $c_X \in \pi_{m+n}(T\widetilde{\nu}_X)$ get mapped to the same element in $\pi_{n+m}(T\gamma^m_{(0)})$. Note that the image of β and c_X, which are $\overline{p}_*^{-1}(p_*\mu) \in H_n(BSO;\mathbb{Z})$ and $\mu \in H_n(X;\mathbb{Z}) \cong \mathbb{Q}$ respectively, get mapped to the same element in $H_n(BSO(m)_{(0)};\mathbb{Z})$. The composite :

$$\pi_{n+m}(T\gamma^m_{(0)}) \to H_{n+m}(T\gamma^m_{(0)};\mathbb{Z}) \to H_n(BSO(m)_{(0)};\mathbb{Z})$$

is a isomorphism, since the Hurewicz map is a isomorphism here. Thus the claim is verified.

We haved proved that the inner-most square of Thom spaces in the main digram is a homotopy cartesian square. Now since β and c_X map to the same element, by the Mayer-Vietoris sequence of the homotopy cartesian square, there exists a class $\alpha \in \pi_{m+n}(T\xi^m)$ such that α gets mapped to c_X. So under the Hurewicz and Thom map, α gets mapped to $pr_{1*}^{-1}\mu \in H_n(PB;\mathbb{Z})$. Using this class α to perform Thom-Pontryagin construction, we obtain a degree 1 normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$ where $g_*[M] = pr_{1*}^{-1}\mu$. Then $f_*[M] = \mu$ for $f = pr_1 \circ g : M \to X$, which completes the proof of Lemma 3.2.2. □

3.3. Surgery obstruction

Given any choice of cohomology classes $p_i \in H^{4i}(X;\mathbb{Q})$, Lemma 3.2.1 constructs a rational nonzero degree normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$. If we specify a choice of fundamental class $\mu \in H_n(X;\mathbb{Q})$ and assume condition (iii) in Theorem 3.1.1, Lemma 3.2.2 constructs a degree 1 normal map such that $g_*[M]$ maps to μ through the localization $pr_1 : PB \to X$. Now we claim that by conditions (i) and (ii) in Theorem 3.1.1, one can perform surgery on the normal map to get a rational homotopy equivalence.

Case 1: $\sigma(X) = 0$

Suppose condition (ii) holds true, i.e. there exists a choice of fundamental class $\mu \in H_{4k}(X;\mathbb{Q})$ such that the intersection form on $H^{2k}(X;\mathbb{Q})$ with respect to μ is isomorphic to $m\langle 1\rangle \oplus m\langle -1\rangle$. Since $\sigma(X) = 0$, the form is hyperbolic with respect to any nonzero fundamental class of X, which is a rational multiple of μ. Then in this case, a rational nonzero degree normal map constructed by Lemma 3.2.1 is sufficient. Since $pr_1 : PB \to X$ is a rational homotopy equivalence, the symmetric form on $H^{2k}(PB;\mathbb{Q})$ with respect to

$g_*[M] \in H_n(PB;\mathbb{Z})$ is also hyperbolic. By condition (i),

$$\begin{aligned} 0 = \langle L_k(p_1,\ldots,p_k), f_*[M]\rangle &= \langle L_k(f^*p_1,\ldots,f^*p_k),[M]\rangle \\ &= \langle L_k(p_1(\tau_M),\ldots,p_k(\tau_M)),[M]\rangle \\ &= \sigma(M) \end{aligned}$$

which implies that the symmetric form on $H^{2k}(M,\mathbb{Q})$ is hyperbolic too. Thus the surgery obstruction vanishes.

Case 2: $\sigma(X) \neq 0$

In this case, we use the candidate degree 1 normal map constructed in Lemma 3.2.2, which has the desired property that $f_*[M] = \mu$. Condition (ii) guaranteed that the intersection form $(H^{2k}(X,\mathbb{Q}),\lambda)$ is contained in the image of the map $W(\mathbb{Z}) \to W(\mathbb{Q})$. In terms of the L-group, condition (ii) guaranteed that the intersection form $(H^{2k}(X,\mathbb{Q}),\lambda)$ has a vanishing $\mathbb{Z}_2$ and $\mathbb{Z}_4$ summands in $L_{4k}(\mathbb{Q})$, so the isomorphism class of the form is solely determined by its signature. By condition (i), we have $\sigma(M) = \sigma(X)$. So the forms $(H^{2k}(M,\mathbb{Q}),\lambda)$ and $(H^{2k}(X,\mathbb{Q}),\lambda)$ are stably isomorphic, hence the surgery obstruction vanishes. □

In dimensions $n \not\equiv 0 \pmod 4$, we have the following version of the realization theorem.

COROLLARY 3.3.1. *For $n \not\equiv 0 \pmod 4$, let X be an n dimensional simply-connected, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex. There always exists an n-dimensional simply-connected smooth closed manifold M which realizes the rational homotopy type of X. From any choice of cohomology classes $p_i \in H^{4i}(X;\mathbb{Q})$, surgery theory constructs a rational homotopy equivalence $f : M \longrightarrow X$ such that $f^*p_i = p_i(\tau_M)$*

PROOF. First notice that when $n \not\equiv 0 \pmod 4$, $L_n(\mathbb{Q}) = 0$, so the surgery obstruction always vanishes. Then the rational nonzero degree normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$ constructed in Lemma 3.2.1 is sufficient to give a rational homotopy equivalence $f = pr_1 \circ g : M \to X$ such that $f^*p_i = p_i(\tau_M)$.

□

3.4. Rational homotopy $\mathbb{CP}^{2n}$

By the rational surgery existence theorem, given any simply-connected $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex X, we can find all the possible Pontryagin numbers of closed smooth manifolds that are rational homotopy equivalent to X. As an application, we will study closed smooth manifolds which are rational homotopy equivalent to $\mathbb{CP}^{2n}$ in terms of their Pontryagin numbers, and compare with the possible Pontryagin numbers of manifolds that are homotopy equivalent to $\mathbb{CP}^{2n}$.

We first construct the localization space $\mathbb{CP}^{2n}_{(0)}$ as in Example 1.2.9. Theorem 3.1.1 states that for any choice of cohomology classes $p_i \in H^{4i}(\mathbb{CP}^{2n};\mathbb{Q})$ and a fundamental class $\mu \in H_{4n}(\mathbb{CP}^{2n}_{(0)};\mathbb{Q}) \cong \mathbb{Q}$ which satisfy:

(i) $\langle L_n(p_1, p_2, \cdots, p_n), \mu\rangle = \pm 1$

(ii) The intersection form $\lambda : H^{2n}(\mathbb{CP}^{2n};\mathbb{Q}) \times H^{2n}(\mathbb{CP}^{2n};\mathbb{Q}) \to \mathbb{Q}$ with respect to μ is isomorphic to a form $m\langle 1\rangle \oplus n\langle -1\rangle$ with m and n integers.

(iii) There exists a closed $4n$ dimensional manifold N such that $\langle p_I(\tau_N), [N] = \langle p_I, \mu\rangle$ for all partitions I of n.

Surgery theory will construct a smooth closed manifold M rational homotopy equivalent to $\mathbb{CP}^{2n}$ with Pontryagin numbers $p_I(\tau_M)[M] = \langle p_I, \mu\rangle$.

3.4.1. Rational homotopy $\mathbb{CP}^4$.

Let α be a generator in $H^2(\mathbb{CP}^4;\mathbb{Z})$ such that $\langle \alpha^2, [\mathbb{CP}^4]\rangle = 1$. Condition (ii) requires a generator $q\alpha^2 \in H^4(\mathbb{CP}^4, \mathbb{Q})$ and a choice of fundamental class $\mu = p\,[\mathbb{CP}^4]$ for some rational numbers p, q, such that $\langle q\alpha^2 \cup q\alpha^2, p[\mathbb{CP}^4]\rangle = q^2 p = \pm 1$, which means $p = \pm\frac{1}{q^2}$. So without loss of generality, we assume $\mu = \pm p^2[\mathbb{CP}^4]$ for some rational number p.

Since the smooth oriented cobordism group $\Omega_8^{SO} \cong \langle \mathbb{CP}^2 \times \mathbb{CP}^2\rangle \oplus \langle \mathbb{CP}^4\rangle$, for any smooth closed 8-dimensional manifold N, there exists $k, l \in \mathbb{Z}$ such that

$$\begin{cases} p_1^2[N] = kp_1^2[\mathbb{CP}^2 \times \mathbb{CP}^2] + lp_1^2[\mathbb{CP}^4] = 18k + 25l \\ p_2[N] = kp_2[\mathbb{CP}^2 \times \mathbb{CP}^2] + lp_2[\mathbb{CP}^4] = 9k + 10l \end{cases}$$

Then condition (iii) requires the existence of $p_1, p_2 \in H^*(\mathbb{CP}^4, \mathbb{Q})$ and $\mu \in H_8(\mathbb{CP}^4, \mathbb{Q})$ such that

$$\begin{cases} 5 \mid \langle p_1^2, \mu\rangle - 2\langle p_2, \mu\rangle \\ 9 \mid 2\langle p_1^2, \mu\rangle - 5\langle p_2, \mu\rangle \\ \langle p_1^2, \mu\rangle \in \mathbb{Z} \\ \langle p_2, \mu\rangle \in \mathbb{Z} \end{cases}$$

We can write $p_1 = a\alpha^2$ for some nonzero rational number a, then $p_1^2 = a^2\alpha^4 \in H^8(\mathbb{CP}^4, \mathbb{Q})$, and $p_2 = b\alpha^4$ for some nonzero rational number b. Then $\langle p_{11}, \mu\rangle = \pm a^2p^2$, $\langle p_2, \mu\rangle = \pm bp^2$. Now let $x^2 = a^2p^2$, $y = bp^2$, then conditions (i),(ii), and (iii) together requires the existence of integers x and y such that:

$$\begin{cases} \langle L_2(p_1, p_2), \mu\rangle = \frac{1}{45}(7y - x^2) = \pm 1 \\ 5 \mid x^2 - 2y \\ 9 \mid 2x^2 - 5y \end{cases}$$

which has infinitely many solutions. Note that the solutions (x^2, y) to the above system of diophantine equations are exactly all possible Pontryagin numbers $(p_{11}[M], p_2[M])$ for a smooth closed manifold that is rational homotopy equivalent to $\mathbb{CP}^4$. Since Pontryagin numbers are homeomorphism invariants, these manifolds fall into infinitely many homeomorphism types.

Remark 3.4.1. In the $\mathbb{CP}^4$ case, the signature condition and the integrality of the Pontryagin numbers ensure the congruence relations, i.e. condition (iii) is automatically true by conditions (i) and (ii). This happens in dimension 8 essentially because the characteristic numbers $s_{11}[\mathbb{CP}^2 \times \mathbb{CP}^2]$ and $s_2[\mathbb{CP}^4]$ are coprime. In the case of $\mathbb{CP}^{2n}$, for $n > 2$, we do not have such simplification.

3.4.2. Rational homotopy $\mathbb{CP}^6$. For $\mathbb{CP}^6$, to satisfy condition (ii), we again assume $\mu = \pm p^2[\mathbb{CP}^6]$ for some rational number p. We can write $p_1 = a\alpha^2$, $p_2 = b\alpha^4$ and $p_3 = c\alpha^6$ for rational numbers a, b and c, then $p_1^3 = a^3\alpha^6$; , $p_1p_2 = ab\alpha^6$, and we have $\langle p_1^3, \mu\rangle = \pm a^3p^2$, $\langle p_1p_2, \mu\rangle = \pm abp^2$ and $\langle p_3, \mu\rangle = \pm cp^2$. To find all possible $p_I[M]$ for M a rational homotopy $\mathbb{CP}^6$, we can assume $p = 1$, then conditions (i),(ii) and (iii) together require the existence of integers a, b and c such that:

$$\begin{cases} \langle L_3(p_1,p_2,p_3),\mu\rangle = \frac{1}{945}(62a^3-13ab+2c)=\pm 1 \\ 27 \mid 7a^3-23ab+28c \\ 15 \mid -6a^3+19ab-21c \\ 7 \mid a^3-3ab+3c \end{cases}$$

In this $\mathbb{CP}^6$ case, the signature condition (i) would not guarantee condition (iii). In dimension 12, the characteristic numbers $s_{1,1,1}[\mathbb{CP}^2\times\mathbb{CP}^2\times\mathbb{CP}^2]$, $s_{1,2}[\mathbb{CP}^2\times\mathbb{CP}^4]$ and $s_3[\mathbb{CP}^6]$ are not coprime.

3.4.3. Comparison between rational homotopy and homotopy $\mathbb{CP}^4$. One can determine the set of Pontryagin numbers for manifolds that are rational homotopy equivalent but not homotopy equivalent to the complex projective space. We will need the following concept of *splitting invariant* and a theorem relating them with the Pontryagin classes.

Definition 3.4.2. [**S3**][**L1**]Let $h: M^{2n}\to\mathbb{CP}^n$ be a degree 1 normal map from a PL manifold M^{2n} to $\mathbb{CP}^n$. One can perturb h within its homotopy class so that h is transverse regular to the submanifold $\mathbb{CP}^k\subset\mathbb{CP}^n$ and $N^{2k}=h^{-1}(\mathbb{CP}^k)$ is a PL manifold for $k=1,\ldots,n-1$. Let

$$\theta_k(N^{2k}\to\mathbb{CP}^k)\in L_{2k}(\mathbb{Z})$$

be the surgery obstruction to make $h|_{N^{2k}}$ normally bordant to a homotopy equivalence. Then the *splitting invariant* is defined to be

$$\sigma_k(h)=\theta_k\in L_{2k}(\mathbb{Z})$$

for $k=1,\ldots,n$

Remark 3.4.3. [**S3**][**L1**] The normal structure set $\mathcal{N}_{PL}(\mathbb{CP}^n)\cong[\mathbb{CP}^n, G/PL]$ is isomorphic to $\prod_{k=2}^n L_{2k}(\mathbb{Z})$, and the PL surgery problems with range $\mathbb{CP}^n$ is determined by the splitting invariants $(\sigma_2,\sigma_3,\ldots,\sigma_n)$. Furthermore all such invariants are realizable.

Let $h : M^{2n} \to \mathbb{CP}^n$ be a homotopy equivalence from a PL manifold M^{2n} to $\mathbb{CP}^n$ with splitting invariants $(\sigma_2, \sigma_3, \ldots, \sigma_n)$ where $\sigma_n = 0$. In [**L1**], Robert D. Little proved the following theorem

THEOREM 3.4.4. [**L1**, Theorem 3.1] *With the above hypothesis, there exist universal polynomials a_i in $\mathbb{Q}[x_1, x_2, \ldots, x_i]$, $i \geq 0$, such that $a_0 = 1$, $a_i(0, 0, \ldots, 0) = 0$, $i \geq 1$, and if $1 \leq i \leq [\frac{1}{2}n]$, the rational Pontryagin classes*

$$p_i(M^{2n}) = \sum_{k=0}^{i} \binom{n+1-k}{i-k} a_k(\sigma_2, \sigma_4, \ldots, \sigma_{2k}) h^* y^{2i}$$

where y is a generator in $H^2(\mathbb{CP}^n; \mathbb{Q})$.

In the proof, the author provided a method of computing the polynomials a_i in the above theorem. In particular,

LEMMA 3.4.5. [**L1**, Lemma 3.4] *The first two polynomials a_i are given by:*

$$a_1(\sigma_2) = 24\sigma_2, \qquad a_2(\sigma_2, \sigma_4) = \frac{1}{7}(360\sigma_4 + 576\sigma_2^2 - 432\sigma_2)$$

Take $\mathbb{CP}^4$ for example. Applying the above theorem and lemma, we have the following relation between the Pontryagin numbers of any PL manifold M^8 with a homotopy equivalence $h : M^8 \to \mathbb{CP}^4$ and the splitting invariant σ_2 of h:

$$\begin{cases} p_1^2[M^8] = (24\sigma_2 + 5)^2 \\ p_2[M^8] = \frac{1}{7}(576\sigma_2^2 + 240\sigma_2 + 70) \end{cases}$$

In [**L2**, Theorem 1.1], the author proved that a PL homotopy $\mathbb{CP}^4$ is smooth if and only if $\sigma_2 \equiv 0 \pmod{14}$ or $6 \pmod{14}$. Plugging-in these two congruences into the above equations, the possible Pontryagin numbers (p_1^2, p_2) for any smooth manifold that is homotopy equivalent to $\mathbb{CP}^4$ are exactly the integers that satisfy

$$\begin{cases} p_1^2 = 25 + 3360n + 112896n^2 \\ p_2 = 10 + 480n + 16128n^2 \end{cases}$$

or

$$\begin{cases} p_1^2 = 22201 + 100128n + 112896n^2 \\ p_2 = 3178 + 14304n + 16128n^2 \end{cases}$$

for any integer n.

By our earlier discussion, the possible Pontryagin numbers for a manifold that is rational homotopy equivalent to $\mathbb{CP}^4$ are exactly those satisfy the signature condition:

$$\langle L_2(p_1, p_2), \mu \rangle = \frac{1}{45}(7p_2 - p_1^2) = \pm 1$$

Comparing the two sets of equations, one can find the Pontryagin numbers that can only be realized by a rational homotopy $\mathbb{CP}^4$ but never a homotopy one. For example, $(p_1^2, p_2) = (4, 7)$ is such a pair of numbers.

One can do similar comparison on higher $\mathbb{CP}^{2n}$'s.

CHAPTER 4

Rational analogs of projective planes

There exist four kinds of projective planes, which are the well-known real, complex, quaternionic and octonionic projective planes. One can prove that there does not exist any higher dimensional projective planes. i.e.

FACT 4.0.1. For $n > 8$, there does not exist any simply-connetced $2n$ dimensional closed manifold M with

$$H^*(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, n, 2n; \\ 0 & \text{otherwise} \end{cases}$$

This fact is a consequence of the well-known Hopf invariant one theorem. Suppose there exists such a manifold M^{2n} for $n > 8$. By [**S**, Theorem 6.1] and [**M2**, Theorem 3.5], there exists a Morse function with the minimal number of critical points which gives a CW complex $X = e^0 \cup e^n \cup_\phi e^{2n}$ that is homotopy equivalent to M. This indicates the existence of a Hopf invariant 1 attaching map $\phi : S^{2n-1} \to S^n$. But the only maps with Hopf invariant 1 are the well known Hopf fibrations $S^{k-1} \hookrightarrow S^{2k-1} \to S^k$ for $k = 1, 2, 4, 8$.

Ignoring the torsion, one can ask the existence of any rational analogs of projective planes in higher dimensions. We will prove the following:

THEOREM 4.0.2. *After dimension 2,4,8,and 16, which are the dimension of $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{OP}^2$ and $\mathbb{HP}^2$, the smallest next dimension where a rational analog of projective plane exists is 32. i.e. there exist 32 dimensional smooth closed manifolds M such that*

$$H^*(M;\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 16, 32; \\ 0 & \textit{otherwise} \end{cases}$$

and such manifolds fall into infinitely many homeomorphism types.

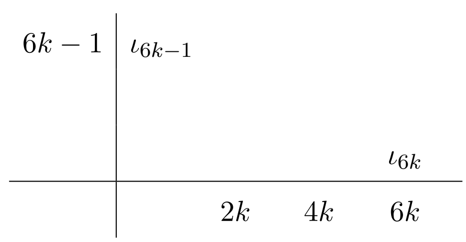
Proof. First, notice that by the desired intersection form, such manifold only exists in dimension $4k$. So we can use Theorem 3.1.1 to study the existence of such manifold in dimension $4k$ for $k > 4$.

We first construct a $4k$ dimensional $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex X, such that X has the desired rational cohomology ring. Consider the following Postnikov tower of rational principal fibration, let $X \to K(\mathbb{Q}, 2k)$ be the principal fibration with fiber $K(\mathbb{Q}, 6k-1)$ and k-invariant ι_{2k}^3, i.e.

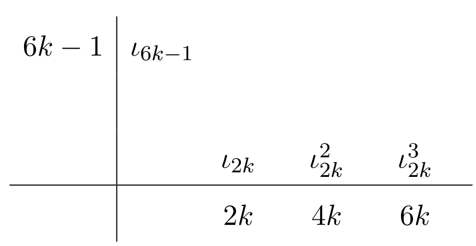
$$\begin{array}{ccc} K(\mathbb{Q},6k-1) & \longrightarrow & K(\mathbb{Q},6k-1) \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \\ \downarrow & & \downarrow \\ K(\mathbb{Q},2k) & \xrightarrow{\iota_{2k}^3} & K(\mathbb{Q},6k) \end{array}$$

The map $\iota_{2k}^3 : K(\mathbb{Q}, 2k) \to K(\mathbb{Q}, 6k)$ induces a morphism between the spectral sequences of the two corresponding fibrations:

$H^p(K(\mathbb{Q}, 6k); H^q(K(\mathbb{Q}, 6k-1); \mathbb{Q})) \Rightarrow H^{p+q}(*; \mathbb{Q})$



$H^p(K(\mathbb{Q}, 2k); H^q(K(\mathbb{Q}, 6k-1); \mathbb{Q})) \Rightarrow H^{p+q}(X; \mathbb{Q})$



In the first spectral sequence, the class ι_{6k-1} is killed by a differential $d^k(\iota_{6k}) = \iota_{6k-1}$, which implies that in the second spectral sequence, ι_{6k-1} is also killed at the stage. So we have:

$$H^*(X;\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,\ 2k,\ 4k; \\ 0 & \text{otherwise} \end{cases}$$

with signature $\sigma(X) = \pm 1$ by construction.

Now we can apply Theorem 3.1.1. For $k > 4$, we require a choice of cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), i = \frac{k}{2}, k$, and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q}) \cong \mathbb{Q}$ such that

(i) $\langle L_k(\cdots, p_{\frac{k}{2}}, \cdots, p_k), \mu\rangle = \pm 1$

(ii) The symmetric form on $H^{2k}(X;\mathbb{Q})$ with respect to μ is isomorphic to $m\langle 1\rangle \oplus n\langle -1\rangle$.

(iii) There exists a closed $4k$ dimensional manifold N such that

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle$$

for all partitions I of k.

When k is odd, the Pontryagin classes p_i is nonzero only when $i = k$. Then condition (i) requires:

$$\langle L_k(0, \cdots, 0, p_k), \mu\rangle = \frac{p}{q}\langle p_k, \mu\rangle = \pm 1$$

where $\frac{p}{q}$ is a fraction with numerator $p \neq \pm 1$ (Remark 1.5.8). At the same time, condition (iii) requires $\langle p_k, \mu\rangle$ to be an integer. So the two conditions can never be both satisfied. Thus there does not exist any simply-connected smooth closed manifold as a rational analog of projective plane in dimension $4k$ with k odd. The next candidate dimension is $4k = 24$.

We will first make condition (iii) explicit enough for computation. The following Hattori-Stong theorem says that the integrality conditions from the Riemann-Roch Theorem together with the integrality of the Pontryagin classes give all the relations on the Pontryagin numbers of smooth closed manifolds. This would provide us a schematic way to compute the congruence relations required by condition (iii).

THEOREM 4.0.3. [**St2**, Theorem 3] *The image of the homomorphism*

$$\tau : \Omega_*^{SO}/tor \to H_*(BSO;\mathbb{Q})$$

is a lattice in $H_*(BSO;\mathbb{Q})$. It consists exactly the elements $x \in H_*(BSO;\mathbb{Q})$ such that:

$$\begin{cases} \langle \mathbb{Z}[e_1, e_2, \cdots] \cdot L, x\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \mathbb{Z}[p_1(\gamma), p_2(\gamma), \cdots], x\rangle \in \mathbb{Z} \end{cases}$$

where e_i is the i-th elementary symmetric function of the variables $e^{x_j} + e^{-x_j} - 2$, and $x_j \in H^2(BSO;\mathbb{Q})$ are the classes in the formal expression of the total Pontryagin class of the universal bundle $p(\gamma) = \Pi(1 + x_j^2)$.

The possible Pontryagin numbers $p_I[M]$ of any smooth closed manifold are exactly the numbers $\langle p_I(\gamma), x\rangle$, where x is contained in the image of the homomorphism

$$\tau : \Omega_*^{SO}/tor \to H_*(BSO;\mathbb{Q})$$

So the integrality conditions in the Hattori-Stong theorem provide exactly the congruence relations required by condition (iii).

In our case, the desired $4k$ dimensional manifold has possibly nonzero Pontryagin classes only in dimension $2k$ and $4k$. We rewrite the cohomology clases e_i in the above integrality conditions as expressions involving only the Pontryagin classes $p_{\frac{k}{2}}$ and p_k, assuming that all the other Pontryagin classes are zero.

In dimension 24, the nonzero e_i classes are

$$\begin{cases} e_1 = \frac{1}{120}p_3 + \frac{1}{79833600}p_3^2 - \frac{1}{39916800}p_6 \\ e_2 = -\frac{1}{4}p_3 + \frac{11}{1209600}p_3^2 + \frac{31}{604800}p_6, \quad e_1e_1 = \frac{1}{14400}p_3^2 \\ e_3 = p_3 + \frac{1}{30240}p_3^2 - \frac{4}{945}p_6, \quad e_1e_2 = -\frac{1}{480}p_3^2, \\ e_4 = \frac{19}{240}p_6, \quad e_1e_3 = \frac{1}{120}p_3^2, \quad e_2e_2 = \frac{1}{16}p_3^2 \\ e_5 = -\frac{1}{2}p_6, \quad e_2e_3 = -\frac{1}{4}p_3^2 \\ e_6 = p_6, \quad e_3e_3 = p_3^2 \end{cases}$$

For the signature condition (i), we can compute, as in Example 1.5.5, the total L class up to dimension 24 as a expression involving only p_3 and p_6:

$$L(0, \ldots, 0, p_3, 0, \ldots, 0, p_6) = 1 + L_3 + L_6 = 1 + \frac{62}{945}p_3 - \frac{40247}{638512875}p_3^2 + \frac{2828954}{638512875}p_6$$

Now we are ready to compute the congruence relations required by condition (iii) explicitly. Plug in the above expressions of the e_i classes into the integrality conditions in the

Hattori-Stong theorem and simplify the coefficients, condition (iii) is equivalent to require a choice of cohomology classes $p_3 \in H^{12}(X;\mathbb{Q}) \cong \mathbb{Q}$, $p_6 \in H^{24}(X;\mathbb{Q}) \cong \mathbb{Q}$ and a fundamental class $\mu \in H_{24}(X;\mathbb{Q})$ such that the following congruence relations hold.

$$\begin{cases} \langle \frac{40247}{638512875}p_3^2 - \frac{2828954}{638512875}p_6, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \frac{43649}{79833600}p_3^2 - \frac{1}{39916800}p_6, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \frac{19829}{1209600}p_3^2 - \frac{31}{604800}p_6, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \frac{397}{6048}p_3^2 + \frac{4}{945}p_6, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \frac{1}{14400}p_3^2, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \frac{19}{240}p_6, \mu\rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle p_3^2, \mu\rangle \in \mathbb{Z} \\ \langle p_6, \mu\rangle \in \mathbb{Z} \end{cases}$$

Let α be a nonzero element in $H^{12}(X,\mathbb{Q}) \cong \mathbb{Q}$, we can write $p_3 = a\alpha$, $p_3^2 = a^2\alpha^2$ and $p_6 = b\alpha^2$ for some nonzero rational number a and b. Let $[X] \in H_{24}(X,\mathbb{Q}) \cong \mathbb{Q}$ be a fundamental class such that $\langle \alpha \cup \alpha, [X]\rangle = 1$. Condition (ii) requires a choice of fundamental class $\mu \in H_{24}(X;\mathbb{Q})$ such that $\mu = \pm p^2[X]$ for some rational number p. Let x and y be the integers such that $x^2 = a^2p^2$, $y = bp^2$, then $\langle p_3^2, \mu\rangle = \pm x^2$, $\langle p_6, \mu\rangle = \pm y$. Then conditions (i),(ii) and (iii) together are equivalent to require the existence of integers x and y such that:

$$\begin{cases} \langle L_6(0,0,p_3,0,0,p_6), \mu\rangle = \pm(\frac{40247}{638512875}x^2 - \frac{2828954}{638512875}y) = \pm 1 \\ 155925 \mid 43649x^2 - 2y \\ 4725 \mid -19829x^2 + 62y \\ 945 \mid 1985x^2 - 128y \\ 225 \mid x^2 \\ 15 \mid y \end{cases}$$

One can compute by hand using quadratic reciprocity or simply use Mathematica to check that the Diophantine equation from the signature condition (i) has no solution in this dimension. So there does not exist any 24 dimensional simly-connected smooth closed manifold such that $H^*(M;\mathbb{Q}) \cong \mathbb{Q}$ for $* = 0, 12, 24$ and zero otherwise. Thus there does not exist any rational analog of projective plane in dimension 24.

REMARK 4.0.4. One can still ask if there exist any 24 dimensional piecewise linear or topological closed manifold which is a rational analog of projective planes. Theorem 3.1.1 still works for the PL or TOP category. And there is a PL version Hattori-Stong theorem charactering the image of the map $\tau : \Omega_*^{PL}/tor \to H_*(BPL;\mathbb{Q})$, which is discussed in [**MM**].

Now we go up to the next candidate dimension, which is 32. The only possible nonzero Pontryagin classes of the desired 32 dimensional manifold are p_4 and p_8. Rewriting the nonzero e_i classes in the Hattori-Stong theorem as expressions involving only p_4 and p_8, we have:

$$\begin{cases} e_1 = -\frac{1}{5040}p_4 + \frac{1}{2615348736000}p_4^2 - \frac{1}{1307674368000}p_8 \\ e_2 = \frac{1}{40}p_4 + \frac{3119}{435891456000}p_4^2 + \frac{5461}{217945728000}p_8, \quad e_1e_1 = \frac{1}{25401600}p_4^2 \\ e_3 = -\frac{1}{3}p_4 + \frac{19}{39916800}p_4^2 - \frac{31}{2851200}p_8, \quad e_1e_2 = -\frac{1}{201600}p_4^2, \\ e_4 = p_4 + \frac{1}{1209600}p_4^2 + \frac{457}{604800}p_8, \quad e_1e_3 = \frac{1}{15120}p_4^2, \quad e_2e_2 = \frac{1}{1600}p_4^2 \\ e_5 = -\frac{43}{2520}p_8, \quad e_1e_4 = -\frac{1}{5040}p_4^2, \quad e_2e_3 = -\frac{1}{120}p_4^2, \\ e_6 = \frac{29}{180}p_6, \quad e_2e_4 = \frac{1}{40}p_4^2, \quad e_3e_3 = \frac{1}{9}p_4^2 \\ e_7 = -\frac{2}{3}p_8, \quad e_3e_4 = -\frac{1}{3}p_4^2 \\ e_8 = p_8, \quad e_4e_4 = p_4^2 \end{cases}$$

For the signature condition (i), we can compute the total L class up to dimension 32 as an expression involving only p_4 and p_8:

$$L(0,\ldots,0,p_4,0,\ldots,0,p_8) = 1+L_4+L_8 = 1+\frac{381}{14175}p_4-\frac{444721}{162820783125}p_4^2+\frac{118518239}{162820783125}p_8$$

Plug in the above expressions of the e_i classes into the integrality conditions in the Hattori-Stong theorem and simplify the coefficients, condition (iii) requires a choice of cohomology classes $p_4 \in H^{16}(X;\mathbb{Q})$, $p_8 \in H^{32}(X;\mathbb{Q})$ and a fundamental class $\mu \in H_{32}(X;\mathbb{Q})$ such that the following congruence relations are satisfied.

$$\begin{cases} \langle 1\cdot L,\mu\rangle = \langle -\frac{444721}{162820783125}p_4^2+\frac{118518239}{162820783125}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_1\cdot L,\mu\rangle = \langle \frac{1992521}{373621248000}p_4^2+\frac{1}{1307674368000}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_2\cdot L,\mu\rangle = \langle \frac{292903727}{435891456000}p_4^2+\frac{5461}{217945728000}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_3\cdot L,\mu\rangle = \langle -\frac{357613}{39916800}p_4^2-\frac{31}{2851200}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_4\cdot L,\mu\rangle = \langle \frac{32513}{1209600}p_4^2+\frac{457}{604800}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_1e_1\cdot L,\mu\rangle = \langle \frac{1}{25401600}p_4^2,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle e_5\cdot L,\mu\rangle = \langle \frac{43}{2520}p_8,\mu\rangle\in\mathbb{Z}[\frac{1}{2}] \\ \langle p_4^2,\mu\rangle\in\mathbb{Z} \\ \langle p_8,\mu\rangle\in\mathbb{Z} \end{cases}$$

Similar as the setup in dimension 24, let x and y be the integers such that $\langle p_4^2,\mu\rangle = \pm x^2$ and $\langle p_8,\mu\rangle = \pm y$. Conditions (i),(ii) and (iii) together require the existence of integers x and y such that:

$$\begin{cases} \langle L_8(0,0,0,p_4,0,0,0,p_8),\mu\rangle = \pm(-\frac{444721}{162820783125}x^2+\frac{118518239}{162820783125}y) = \pm 1 \\ 638512875 \mid 13947647x^2+2y \\ 212837625 \mid 292903727x^2+10922y \\ 155925 \mid 357613x^2+434y \\ 4725 \mid 32513x^2+914y \\ 99225 \mid x^2 \\ 315 \mid y. \end{cases}$$

One can compute by hand using quadratic reciprocity or simply use Mathematica to check that the above system of Diophantine equations has infinitely many solutions. For example,

$$x = 493965360,\ y = 915578185531275.$$

is one pair of solution. Note that distinct solutions of integers x and y correspond to distinct pairs of Pontryagin numbers of the realizing manifold. Since Pontryagin numbers are homeomorphism invariants, the resulting 32 dimensional smooth closed manifolds which are rational analogs of projective planes fall into infinitely many homeomorphism types. □

REMARK 4.0.5. As another approach to compute the congruence relations satisfied by the Pontryagin numbers of all the smooth closed manifolds in dimension $4k$, we can compute the Pontryagin numbers of a set of manifolds as a basis of the torsion-free part of the $4k$ dimensional oriented cobordism group.

The torsion free part of the oriented cobordism ring is a polynomial ring generated by a set of smooth closed manifolds in dimension $4k$'s:

$$\Omega_*^{SO}/tor \cong \mathbb{Z}[M^4, M^8, \ldots]$$

where the generator M^{4k} can be taken to be any manifold satisfying the following characteristic number property [**St1**]:

$$s_k(p_1, \ldots, p_k)[M^{4k}] = \begin{cases} \pm q & \text{if } 2k+1 \text{ is a power of the prime } q; \\ \pm 1 & \text{if } 2k+1 \text{ is not a prime power} \end{cases}$$

Pontryagin numbers are oriented cobordism invariants. If we obtain the Pontryagin numbers of a set of manifolds which is a basis of Ω_{4k}^{SO}/tor, condition (iii) can be rewritten explicitly as a set of congruence relations. Since $s_k[\mathbb{CP}^{2k}] = 2k+1$, in many of the $4k$ dimensions (when $2k+1 = q$ with q a prime), $\mathbb{CP}^{2k}$ qualifies as a generator. For example, in dimension 8,

$$\Omega_8^{SO} \cong \langle \mathbb{CP}^2 \times \mathbb{CP}^2 \rangle \oplus \langle \mathbb{CP}^4 \rangle$$

For any smooth closed 8-dimensional manifold N, each Pontryagin number of N is a linear combination of the corresponding Pontryagin number of $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$, i.e.

$$\begin{cases} p_{11}[N] = kp_{1,1}[\mathbb{CP}^2 \times \mathbb{CP}^2] + \ell p_{1,1}[\mathbb{CP}^4] = 18k + 25\ell \\ p_2[N] = kp_2[\mathbb{CP}^2 \times \mathbb{CP}^2] + \ell p_2[\mathbb{CP}^4] = 9k + 10\ell \end{cases}$$

with $k, \ell \in \mathbb{Z}$. So in this dimension, condition (iii) in Theorem 3.1.1 requires a choice of $p_1, p_2 \in H^*(X, \mathbb{Q})$ and $\mu \in H_8(X_{(0)}; \mathbb{Q})$ such that the following congruence relations hold:

$$\begin{cases} 5 \mid \langle p_{1,1}, \mu \rangle - 2\langle p_2, \mu \rangle \\ 9 \mid 2\langle p_{1,1}, \mu \rangle - 5\langle p_2, \mu \rangle \\ \langle p_{1,1}, \mu \rangle \in \mathbb{Z} \\ \langle p_2, \mu \rangle \in \mathbb{Z} \end{cases}$$

In dimensions such as $4k = 16$ and $4k = 28$, $\mathbb{CP}^{2k}$ does not satisfy the characteristic number property, so the complex projective plane does not qualify as a generator in these dimensions. In fact, the disjoint union of $\mathbb{CP}^{2k}$ and certain complex hypersurfaces can be taken as the generator in these dimensions .

LEMMA 4.0.6. [**M1**, Part 4] *If $m, n > 1$, then for $\mathcal{H}_{m,n}$ as hypersurface of degree $(1,1)$ in $\mathbb{CP}^m \times \mathbb{CP}^n$*

$$s_{2k}(c)[\mathcal{H}_{m,n}] = -\frac{(m+n)!}{m!n!}$$

where $s_{2k}(c)$ is the s_{2k} characteristic number of the Chern classes.

We also have the following lemma relating the Chern classes and the Pontryagin classes:

LEMMA 4.0.7. [**MS**, Problem 16-C] *If $2i_1, \ldots, 2i_r$ is a partition of $2k$ into even integers, the $4k$-dimensional characteristic class $s_{2i_1,\ldots,2i_r}(c(\omega))$ of a complex vector bundle ω is equal to the characteristic class $s_{i_1,\ldots,i_r}(p(\omega_{\mathbb{R}}))$ of its underlying real vector bundle. In particular,*

$$s_{2k}(c(\omega)) = s_k(p(\omega_{\mathbb{R}}))$$

Then for fixed $2k+1 = m+n$, one can compute the desired characteristic number of all the complex hypersurfaces $\mathcal{H}_{m,n}$:

$$s_k(p)[\mathcal{H}_{m,n}] = -\frac{(m+n)!}{m!n!} = -\binom{m+n}{m} = -\binom{2k+1}{m}$$

On the other hand, the greatest common divisor of the numbers $\binom{2k+1}{m}$ for $1 \leq m \leq k$ is 1 when $2k+1$ is not a prime power [**M1**, Part 4]. So for each $4k$ dimension, we can choose the generator of the cobordism ring to be a disjoint union of certain complex hypersurfaces (possibly including $\mathbb{CP}^{2k}$) so that it has the desired s_k number.

For example, in dimension $4k = 16$

$$s_4(p)[9\mathbb{CP}^8 + \mathcal{H}_{3,6}] = -3$$

and in dimension $4k = 28$

$$s_7(p)[-85\mathbb{CP}^{14} - 16\mathcal{H}_{3,12} + 2\mathcal{H}_{5,10}] = -1$$

After we obtain the generating manifolds of the cobordism ring in each candidate dimension $4k$, we will need to compute all the Pontryagin numbers p_I for a set of basis manifolds to get the congruence relations in the candidate dimension. This is a huge computation.

CHAPTER 5

Non-simply-connected rational surgery

We prove the rational surgery existence theorem in the case where the fundamental group of the starting local space is a finite group. We apply this theorem to study the realization problems for certain family of rational cohomology algebras.

5.1. Existence theorem

THEOREM 5.1.1. *Given an $n = 4k$ dimensional, $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex X, with $\pi_1(X) = \pi$ a finite group and trivial orientation character ω, there exists a smooth closed $4k$ dimensional manifold M, and a $\mathbb{Q}$-homotopy equivalence $f : M \longrightarrow X$ such that $f^*p_i = p_i(\tau_M)$ if and only if:*

Case 1*: The signature $\sigma(X) = 0$*

There exists cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \leq i \leq k$ and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q})$ such that:

(i) $L_k(p_1, \ldots, p_k) = 0 \in H^{4k}(X;\mathbb{Q})$

(ii) *The intersection form $H^{2k}(\widetilde{X};\mathbb{Q}) \times H^{2k}(\widetilde{X};\mathbb{Q}) \to \mathbb{Q}$ defined as $\langle \cdot \cup \cdot, \tau_*\mu\rangle$ admits an π-invariant lagrangian, where $\tau_* : H_*(X;\mathbb{Q}) \to H_*(\widetilde{X};\mathbb{Q})$ is the homology transfer homomorphism.*

(iii) $\forall g \in \pi_1(X) - \{e\}$,

$$\sum (-1)^i (tr(g_* : H_i(\widetilde{X};\mathbb{Q}) \to H_i(\widetilde{X};\mathbb{Q}))) = 0$$

Case 2*: The signature $\sigma(X) \neq 0$*

There exists cohomology classes $p_i \in H^{4i}(X;\mathbb{Q}), 1 \leq i \leq k$, and a fundamental class $\mu \in H_{4k}(X;\mathbb{Q}) \cong \mathbb{Q}$ such that

(i) $\langle L_k(p_1, \ldots, p_k), \mu\rangle = \sigma(X)$

(ii) *For the intersection form* $\lambda : H^{2k}(\widetilde{X};\mathbb{Q}) \times H^{2k}(\widetilde{X};\mathbb{Q}) \to \mathbb{Q}$ *defined as* $\langle \cdot \cup \cdot, \tau_* \mu\rangle$ *is the homology transfer homomorphism,* $(H^{2k}(\widetilde{X};\mathbb{Q}), \lambda) \oplus \sigma(X)\langle -1\rangle$ *admits an invariant lagrangian, where the form* $\langle -1\rangle : \mathbb{Q}\pi \times \mathbb{Q}\pi \to \mathbb{Q}$ *is defined as* $(a, b) \mapsto -tr_e(b\overline{a})$*.*

(iii) *There exists a closed smooth* $4k$ *dimensional manifold* N *such that*

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle$$

for all partitions I *of* k*.*

(iiii) $\forall g \in \pi_1(X) - e$,

$$\sum (-1)^i (\mathrm{tr}(g_* : H_i(\widetilde{X};\mathbb{Q}) \to H_i(\widetilde{X};\mathbb{Q}))) = 0$$

If the choice of cohomology classes p_i *and fundamental class* μ *satisfy all the above conditions* ((i), (ii) *and* (iii) *in case 1;* (i),(ii), (iii) *and* (iiii) *in case 2*), surgery theory will construct a* $\mathbb{Q}$*-homotopy equivalence* $f : M \to X$ *which satisfies* $f^* p_i = p_i(\tau_M)$*, where* $p_i(\tau_M)$ *are the Pontryagin classes of the tangent bundle of* M*, and in case 2, the Pontryagin numbers* $p_I[M] = \langle p_I, \mu\rangle$ *for all partitions* I *of* k*.*

Proof. ($\Longrightarrow$) Assuming there exists a smooth closed manifold M realizing the rational homotopy type of X, one can check that conditions (i),(ii) and (iii) hold true. By Lemma 2.1, condition (iiii) is a necessary and sufficient condition for X to have a rational homotopy type of a finite CW complex.

($\Longleftarrow$):

5.1.1. Rational degree 1 normal map. The proof of the existence of rational degree 1 normal map is similar to the simply-connected case. But instead we begin with the map

$$p \times u : X \to BSO(m)_{(0)} \times B\pi$$

where $u : X \to B\pi$ is the classifying map for the universal cover $\widetilde{X}$. And we obtain the following diagram similar to the one in the simply-connected case.

If we are in the case that $\sigma(X) = 0$, a nonzero degree normal map is enough. Similar to the simply-connected case, we can find a class $\alpha \in \pi_{n+k}(T\xi^m)$ which maps to a nonzero class in $H_n(X;\mathbb{Q})$. So we can perform the Thom-Pontryagin construction to get a nonzero degree normal map.

$$\begin{array}{ccccc}
\nu_M & \longrightarrow & \xi & \longrightarrow & \gamma^{m-1} \times \epsilon \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{g} & PB & \xrightarrow{pr_2} & BSO(m) \times B\pi \\
& \searrow^{f} & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \overline{p}} \\
& & X & \xrightarrow{p\times u} & BSO(m)_{(0)} \times B\pi
\end{array}$$

In the case of $\sigma(X) \neq 0$, condition (iii) guarantees the existence of a rational degree 1 normal map. The proof is similar to the simply-connected case.

There exists a closed manifold N such that for all partitions I of k

$$\langle p_I(\tau_N), [N]\rangle = \langle p_I, \mu\rangle = \langle \overline{p}_I, \overline{p}_*^{-1}((p \times u)_*\mu)\rangle$$

which implies $\langle p_I(\nu_N), [N]\rangle = \langle p_I(\gamma^m), \overline{p}_*^{-1}((p \times u)_*\mu)\rangle$. Now since we assumed π is finite,

$$H^*(BSO \times B\pi; \mathbb{Q}) \cong H^*(BSO; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots]$$

Then since $H_n(BSO(m) \times B\pi; \mathbb{Q}) \cong \mathrm{Hom}(H^n(BSO(m) \times B\pi; \mathbb{Q}), \mathbb{Q})$, the condition implies that $\overline{p}_*^{-1}((p \times u)_*\mu)$ lies in the image of the homomorphism

$$\nu : \Omega_n^{SO}(B\pi) \to H_n(BSO \times B\pi; \mathbb{Q})$$

where for $(M,\ f{:}M \to B\pi) \in \Omega_n^{SO}(B\pi)$, $\nu(M) = (\nu_M \times f)_*[M]$ where ν_M is the classifying map for the normal bundle of M. Here ϵ is the trivial 1-bundle over $B\pi$.

$$\begin{array}{ccc}
\nu_M & \longrightarrow & \gamma \times \epsilon \\
\downarrow & & \downarrow \\
M & \xrightarrow{\nu_M \times f} & BSO \times B\pi
\end{array}$$

Since $T(\gamma \times \epsilon) = T\gamma \wedge T\epsilon = T\gamma \wedge SB\pi_+$

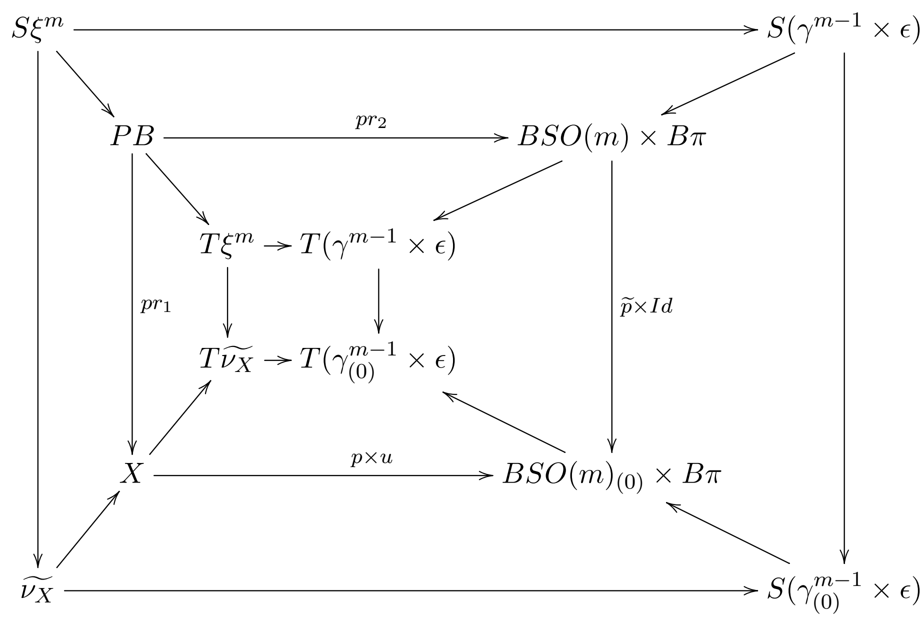
$$\begin{aligned}
\Omega_n^{SO}(B\pi) &\cong \lim_{m\to\infty} \pi_{n+m}(T\gamma^m \wedge B\pi_+) \\
&\cong \lim_{m\to\infty} \pi_{n+m}(ST\gamma^{m-1} \wedge B\pi_+) \\
&\cong \lim_{m\to\infty} \pi_{n+m}(T\gamma^{m-1} \wedge SB\pi_+) \\
&\cong \lim_{m\to\infty} \pi_{n+m}(T(\gamma^{m-1} \times \epsilon))
\end{aligned}$$

Hence ν can also be interpreted as the map ν in the following diagram:

$$\begin{array}{ccc}
\Omega_n^{SO}(B\pi) \cong \lim_{m\to\infty} \pi_{n+m}(T(\gamma^{m-1}\times\epsilon)) & \xrightarrow{h} & \lim_{m\to\infty} H_{n+m}(T(\gamma^{m-1}\times\epsilon);\mathbb{Z}) \\
\Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \cap U} \\
H_n(BSO\times B\pi;\mathbb{Q}) & \xrightarrow{i_*} & H_n(BSO\times B\pi;\mathbb{Z})
\end{array}$$

Now since $\overline{p}_*^{-1}((p\times u)_*\mu) \in \mathrm{Im}\ \nu$, there exists a class $\beta \in \pi_{n+m}(T(\gamma^{m-1}\times\epsilon))$ such that $\nu(\beta) = i_*(h(\beta)\cap U) = \overline{p}_*^{-1}((p\times u)_*\mu) \in H_n(BSO\times B\pi;\mathbb{Z})$.

Similar to the simply-connected case, we can use Lemma 5.1 and Lemma 1.4 to prove that the square of Thom spaces in the following diagram is a homotopy cartesian square.



We can prove that there exists a class $\alpha \in \pi_{m+n}(T\xi^m)$ such that α gets mapped to ${pr_1}_*^{-1}\mu \in H_n(PB;\mathbb{Z})$ under the Hurewicz and Thom map. Using this class α to perform Thom-Pontryagin construction, we get a rational degree 1 normal map $(g,\hat{g}) : (M,\nu_M) \to (PB,\xi)$ where $g_*[M] = {pr_1}_*^{-1}\mu$, and then $f_*[M] = \mu$ for $f = pr_1 \circ g : M \to X$.

5.1.2. Surgery obstruction. For the surgery obstruction part in the non-simply-connected case, we need the following theorem relating the signature of a manifold to its rational equivariant intersection form.

THEOREM 5.1.2. [**D**, Theorem A] *Let M^{2k} be a free (G,ω)-manifold. Then the intersection form on $H^k(M,\mathbb{Q})$ has an invariant Lagrangian if*

(a) $\omega = 1$ *and* $\sigma(M/G) = 0$*, or*

(b) $\omega \neq 1$ *and* $\chi(M/G)$ *is even.*

In the case $\sigma(X) = 0$, condition (i) in Theorem 5.1.1 implies $\sigma(M) = \sigma(X) = 0$. Then we apply Theorem 5.1.2 to the universal cover $\widetilde{M}$, so the intersection form on $H^{2k}(\widetilde{M},\mathbb{Q})$ admits a Lagrangian. By condition (ii) in the main theorem, the intersection form on $H^{2k}(\widetilde{X},\mathbb{Q})$ also admits a Lagrangian, so the rational intersection form on $\widetilde{M}$ and $\widetilde{X}$ are isomorphic. Hence the surgery obstruction vanishes.

In the case $\sigma(X)=m \neq 0$, condition (i) in Theorem 5.1.1 implies $\sigma(M) = m$. Then we have $\sigma(M\#m\overline{\mathbb{CP}^2}) = 0$. By Theorem 5.1.2, the intersection form $(H^{2k}(\widetilde{M},\mathbb{Q}),\lambda) \oplus m\langle -1\rangle$ admits a Lagrangian. By condition (ii) in Theorem 5.1.1, $(H^{2k}(\widetilde{X},\mathbb{Q}),\lambda) \oplus m\langle -1\rangle$ also admits a Lagrangian. Then by Witt cancelation, the rational intersection form on $\widetilde{M}$ and $\widetilde{X}$ are isomorphic. Hence the surgery obstruction vanishes.

In the case that $\sigma(X) \neq 0$, condition (ii) on the equivariant form can be verified directly using the G-signature Theorem.

THEOREM 5.1.3. [**AS**, Atiyah-Singer G-signature Theorem] *Let M be a closed oriented manifold of dimension $4k$ with a G-action. The intersection form*

$$\lambda : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to \mathbb{R}$$

is a nonsingular G-invariant bilinear form.There exists a G-invariant decomposition

$$H^{2k}(M;\mathbb{R}) = V_+ \oplus V_-$$

where V_+ (V_-) are the G-invariant positive (negative) definite subspaces of the symmetric bilinear form λ. The G-signature is defined to be

$$\sigma_G(M) = [V_+] - [V_-] \in \widetilde{K_0}(\mathbb{R}G)$$

If the G-action is free, $\sigma_G(M) = 0$.

EXAMPLE 5.1.4. When $G = \mathbb{Z}_2 = \langle t \rangle$,

$$\mathbb{Q}[\mathbb{Z}_2] \cong \mathbb{Q}_+ \times \mathbb{Q}_-$$

Let

$$\lambda_\pm : H^{2k}(\widetilde{X};\mathbb{Q})_\pm \times H^{2k}(\widetilde{X};\mathbb{Q})_\pm \to \mathbb{Q}$$

be the restriction of the intersection form λ on the two invariant subspaces respectively. Then condition (ii) in Theorem 5.1.1 is equivalent to requiring that

(a) Each of the intersection forms λ_+ and λ_- is isomorphic to $m\langle 1\rangle \oplus n\langle -1\rangle$.

(b) The G-signature

$$\sigma_{\mathbb{Z}_2}(X) = \text{signature}(\lambda_+)[\mathbb{Q}_+] + \text{signature}(\lambda_+)[\mathbb{Q}_-] = 0 \in \widetilde{K_0}(\mathbb{Q}[\mathbb{Z}_2])$$

□

5.2. Applications

The non-simply-connected version of the rational surgery existence theorem can be used to study the following realization question:

Given a $\mathbb{Q}$-Poincaré duality algebra Λ^* and a G-action on Λ^*, does there exist a closed smooth manifold M with $\pi_1(M) = G$ and $H^*(\widetilde{M};\mathbb{Q}) \cong \Lambda^*$? This is equivalent to asking if there exist a free G-action on a smooth closed manifold whose cohomology ring realizes the given algebra with the action?

For G finite with the action preserving the orientation, we can apply Theorem 5.1.1 to answer the above question as follows:

Step 1: Construct a simply-connected $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex Y with the specified G-action on its cohomology ring $H^*(Y;\mathbb{Q}) \cong \Lambda^*$.

Step 2: Let the starting local space X in Theorem 5.1.1 be the homotopy orbit space of Y, i.e. $X = (EG \times Y)/G$. Then X is a $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex with $\pi_1(X) = G$ and G acts freely on $\widetilde{X}$ with $H^*(\widetilde{X};\mathbb{Q}) \cong \Lambda^*$.

Step 3: Check the conditions in Theorem 5.1.1.

Remark 5.2.1. It is possible that an algebra Λ^* can be realized by more than one rational homotopy type. There is a special family of simply-connected commutative graded algebras, called *intrinsically formal* algebras, which can be realized by exactly one rational homotopy type [**FH**], i.e. if a simply-connected commutative graded algebra Λ^* is *intrinsically formal*, then for any two simply-connected spaces X and Y such that $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q}) \cong \Lambda^*$, X and Y are rational homotopy equivalent to each other.

We will study the realization question on a special family of rational Poincaré duality algebras ($\mathbb{Q}$-PDA), which are called homogeneous artinian complete intersections.

Definition 5.2.2. A homogeneous artinian complete intersection $\mathcal{A}$ is a commutative graded $\mathbb{Q}$-algebra of the form

$$\mathcal{A} = \mathbb{Q}[x_1, x_2, \ldots . x_n]/\mathcal{I}$$

where the variables x_i have positive even degree $|x_i|$, and the ideal $\mathcal{I}$ is generated by the regular sequence

$$\mathcal{I} = (f_1, f_2, \ldots, f_n)$$

where f_i are homogeneous polynomials of degree $|f_i| = 2d_i$. Such a algebra $\mathcal{A}^*$ is a 1-connected rational Poincaré duality algebra [**FH**], and it has formal dimension

$$m = \sum_{i=1}^{n}(|f_i| - |x_i|)$$

Definition 5.2.3. Here a sequence $f_1, f_2, \ldots, f_n$ is called a *regular sequence* if for each $i = 1, 2, \ldots, n$, f_i is not a zero divisor in $\mathbb{Q}[x_1, x_2, \ldots, x_n]/(f_1, \ldots, f_{i-1})$

Example 5.2.4.

$$\mathcal{A}^* = \mathbb{Q}[x, y]/(x^2 + y^2, x^3 y)$$

with $|x| = 2; |y| = 2$ is a $\mathbb{Q}$-Poincaré duality algebra of formal dimension

$$m = 4 - 2 + 8 - 2 = 8$$

REMARK 5.2.5. [**FH**] Let $\Lambda(V)$ be a free, graded commutative algebra over a graded $\mathbb{Q}$ vector space V, i.e. a tensor product of exterior algebra on odd generators and the polynomial algebra on even generators. Let $\mathcal{I}$ be an ideal generated by a regular sequence. Then any algebra of the form

$$\Lambda(V)/\mathcal{I}$$

is intrinsically formal. In particular, any homogeneous artinian complete intersection $\mathcal{A}^*$ is intrinsically formal. So once we construct a local space Y whose rational cohomology ring realizes $\mathcal{A}^*$, Y also carries the unique rational homotopy data.

In [**PL**], the authors studied the homogeneous artinian complete intersections of formal dimension 8. They classified such 8 dimensional $\mathbb{Q}$-PDAs into 5 different cases in terms of different sets of $|f_i|$ values (with all the possible values of the signature specified)

$$(I_8) \qquad (|f_1|, |f_2|, |f_3|, |f_4|) = (4, 4, 4, 4); |\sigma| = 0, 2, 4, 6$$

$$(II_8) \qquad (|f_1|, |f_2|, |f_3|) = (4, 4, 6); |\sigma| = 0, 2, 4$$

$$(III_8) \qquad (|f_1|, |f_2|) = (4, 8); |\sigma| = 0, 2$$

$$(IV_8) \qquad (|f_1|, |f_2|) = (6, 6); |\sigma| = 1, 3$$

$$(V_8) \qquad (|f_1|) = (10); |\sigma| = 1$$

and proved that in each of the cases, all the possible signature values can be realized by smooth closed manifolds, i.e. for each case above and each possible signtaure value σ, there exists a smooth closed manifold M such that $H^*(M;\mathbb{Q}) \cong \mathcal{A}^*$ where $\mathcal{A}^*$ belongs to the case and $\sigma(M) = \sigma$.

We will now study the realization question in the non-simply-connected case. In particular, we study the question addressed at the beginning of the section for $\mathbb{Q}$-PDAs belonging to the case (III_8) above, i.e. of the form

$$\mathcal{A}^* = \mathbb{Q}[x,y]/(f_1, f_2)$$

with $|x| = |y| = 2$ and $|f_1| = 4, |f_2| = 8$. We will find all the possible finite G-actions on $\mathcal{A}^*$ such that there exists a free G-action on a smooth closed manifold whose cohomology ring realizes $\mathcal{A}^*$ with the specified action. As mentioned at the beginning of this section, this is equivalent to asking for the existence of smooth closed manifold M such that $\pi_1(M) = G$ and $H^*(\widetilde{M};\mathbb{Q}) \cong \mathcal{A}^*$. This is a question that can be studied using the non-simply-connected rational surgery existence theorem (Theorem 5.1.1).

One can check by direct computation that any $\mathbb{Q}$-PDA of the form (III_8) falls into one of the following three different cases up to algebra isomorphism:

$$\begin{array}{ll} (i) & f_1 = xy, f_2 = kx^4 - y^4 \quad (k \neq 0) \\ (ii) & f_1 = y^2, f_2 = x^4 \\ (iii) & f_1 = kx^2 - y^2, f_2 = x^4 \ \text{ or } \ f_2 = \ell x^4 - x^3 y \quad (k \neq 0) \end{array}$$

For each case, we will first check all the possible actions of finite groups on $\mathcal{A}^*$ such that the Lefschetz fixed point condition is satisfied. This determines the candidate finite groups G and the G-action on $\mathcal{A}^*$. Then we follow the three steps mentioned at the beginning of the section.

Case (i): $\mathcal{A}^* = \mathbb{Q}[x,y]/(xy, kx^4 - y^4)$ with $k \neq 0$.

First notice that when $k \equiv \pm 1 \pmod{\mathbb{Q}^{*4}}$, $\mathcal{A}^*$ is isomorphic to

$$\begin{aligned} H^*(\mathbb{CP}^4 \# \mathbb{CP}^4;\mathbb{Q}) &\cong \mathbb{Q}[x,y]/(xy, x^4 - y^4) \ \text{ or} \\ H^*(\mathbb{CP}^4 \# \overline{\mathbb{CP}^4};\mathbb{Q}) &\cong \mathbb{Q}[x,y]/(xy, x^4 + y^4) \end{aligned}$$

For any g of finite order that acts on $\mathcal{A}^*$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathcal{A}^2 \cong \mathbb{Q}x \oplus \mathbb{Q}y$

$$\begin{cases} g \cdot xy = 0 \\ g \cdot kx^4 = g \cdot y^4 \end{cases}$$

which implies that

$$g = \begin{pmatrix} \pm 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{when} \quad k \not\equiv \pm 1 \pmod{\mathbb{Q}^{*4}}$$

$$g = \begin{pmatrix} \pm 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{when} \quad k \equiv \pm 1 \pmod{\mathbb{Q}^{*4}}$$

Moreover, if we require the Lefschetz fixed point condition, the only possible nontrivial finite action with Lefschetz number zero is

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So $G = \mathbb{Z}_2$, generated by $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, is the only finite group that could possibly act freely on a manifold whose cohomology ring realizes such $\mathcal{A}^*$.

Then we construct a simply-connected $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex Y by a three-stage Postnikov tower, realizing the desired cohomology ring $\mathcal{A}^*$. First let $Z \to K(\mathbb{Q},2) \times K(\mathbb{Q},2)$ be the principal fibration with fiber $K(\mathbb{Q},3)$ and k-invariant $\iota_x \iota_y \in H^4(K(\mathbb{Q},2) \times K(\mathbb{Q},2);\mathbb{Q})$; this will kill the class xy in $\mathbb{Q}[x,y]$. Let $Y \to Z$ be the principal fibration with fiber $K(\mathbb{Q},7)$ and k-invariant $k\iota_x^4 - \iota_y^4 \in H^8(Z;\mathbb{Q})$. Then we have $H^*(Y;\mathbb{Q}) \cong \mathbb{Q}[x,y]/(xy, kx^4 - y^4)$.

Now we can use the general fact [**FHT**, Section 17] that there exists a *spatial realization functor* from the category of differential graded algebras to the category of CW complexes. In our case, if we plug-in a $\mathbb{Q}$-PDA $\mathcal{A}^*$ with a G-action on it, the functor gives us a simply-connected $\mathbb{Q}$-local space Y with $H^*(X) \cong \mathcal{A}^*$ and a G-action on Y realizing the G-action on $\mathcal{A}^*$.

Let the starting local space X in Theorem 5.1.1 be the homotopy orbit space of Y. i.e. $X = (E\mathbb{Z}_2 \times Y)/\mathbb{Z}_2$. Then X is a $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex with $\pi_1(X) = \mathbb{Z}_2$ and $H^*(\widetilde{X};\mathbb{Q}) \cong \mathcal{A}^*$. By the property of the transfer homomorphism which says that for $\pi_1(X) = G$, $H^*(X;\mathbb{Q}) \cong H^*(\widetilde{X};\mathbb{Q})^G$, we can compute that

$$H^*(X;\mathbb{Q}) \cong \mathbb{Q}[\alpha,\beta]/(\alpha\beta, k\alpha^2 - \beta^2)$$

with $|\alpha| = |\beta| = 4$, where α, β pull back to x^2 and y^2 in $H^4(\widetilde{X};\mathbb{Q})$.

Now we are ready to determine whether there exists a smooth closed manifold M realizing the rational homotopy type of X by checking the conditions in Theorem 5.1.1. Let

$\mu \in H_8(X;\mathbb{Q})$ be the fundamental class such that $\langle \alpha^2, \mu\rangle = 1$, then the intersection form is

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

with signature $\sigma = 0$ or 2. It is isomorphic to $\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ if and only if $k \equiv \pm 1 \pmod{\mathbb{Q}^{*2}}$

If the signature $\sigma = 0$, for condition (i) in Theorem 5.1.1, we can pick $p_1 = 0 \in H^4(X;\mathbb{Q})$ and $p_2 = 0 \in H^8(X;\mathbb{Q})$, so

$$L_2(p_1, p_2) = 0$$

for condition (ii), the equivariant intersection form $H^4(\widetilde{X};\mathbb{Q}) \times H^4(\widetilde{X};\mathbb{Q}) \to \mathbb{Q}$ defined as $\langle \cdot \cup \cdot, x^3 y\rangle$ is hyperbolic if and only if

$$k \equiv -1 \pmod{\mathbb{Q}^{*2}}$$

If the signature $\sigma = 2$, one can compute that the G-signature $\sigma_{\mathbb{Z}_2}(\widetilde{X}) \neq 0 \in \widetilde{K}_0(\mathbb{Q}[\mathbb{Z}_2])$, so condition (ii) can not be satisfied. We have proved the following:

Theorem 5.2.6. *For any $\mathbb{Q}$-PDA $\mathcal{A}^*$ of the form in Case (i), there exists a free $\mathbb{Z}_2$-action on a smooth closed manifold whose rational cohomology ring realizes $\mathcal{A}^*$ with the action $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathcal{A}^2$. And this $\mathbb{Z}_2$-action is the only possible finite action on $\mathcal{A}^*$ that can be realized by a free orientation-preserving action on a manifold whose cohomology ring realizes $\mathcal{A}^*$.*

Case (ii): $\mathcal{A}^* = \mathbb{Q}[x,y]/(y^2, x^4)$

First we notice that

$$\mathbb{Q}[x,y]/(y^2, x^4) \cong H^*(\mathbb{CP}^3 \times S^2;\mathbb{Q})$$

so any manifold whose cohomology realizes such $\mathcal{A}^*$ is rational homotopy equivalent to $\mathbb{CP}^3 \times S^2$.

As in Case (i), one can check that

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the only possible finite action on $\mathcal{A}^*$ with Lefschetz number zero. So $G = \mathbb{Z}_2$ is the only finite group that could possibly act freely on a manifold whose cohomology ring realizes such $\mathcal{A}^*$.

Notice that on $\mathbb{CP}^3$, the action

$$[x : y : z : t] \mapsto [-\overline{y} : \overline{x} : -\overline{t} : \overline{z}]$$

gives a free action of order 2, together with the antipodal map on S^2, this free $\mathbb{Z}_2$-action on $\mathbb{CP}^3 \times S^2$ realized the $\mathbb{Z}_2$-action on the cohomology ring $\mathcal{A}^*$ with generator $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

But we can still ask, does there exist a smooth closed manifold M such that there exists a free $\mathbb{Z}_2$-action on M that realizes the action above on $H^*(M;\mathbb{Q}) \cong \mathcal{A}^*$ but M is not homeomorphic to $\mathbb{CP}^3 \times S^2$? The answer is yes, and there exist manifolds which are not obatined from the obvious construction of connecting sum with a rational homology sphere.

We follow the same procedure as in Case (i), constructing the local space Y which agrees with the localization of $\mathbb{CP}^3 \times S^2$ and a corresponding $\mathbb{Z}_2$-action on it. Then we obtain the starting local space $X = (E\mathbb{Z}_2 \times Y)/\mathbb{Z}_2$, with $\pi_1(X) = \mathbb{Z}_2$, $H^*(\widetilde{X};\mathbb{Q}) \cong \mathcal{A}^*$, and by the transfer homomorphism,

$$H^*(X;\mathbb{Q}) \cong \mathbb{Q}[\alpha, \beta]/(\alpha^2, \beta^2)$$

with $|\alpha| = |\beta| = 4$, where α, β pull back to x^2 and xy in $H^4(\widetilde{X};\mathbb{Q})$. Let $\mu \in H_8(X;\mathbb{Q})$ be the fundamental class such that $\langle \alpha\beta, \mu \rangle = 1$, then the intersection form is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with signature $\sigma = 0$.

For condition (i) in Theorem 5.1.1, any Pontryagin class $p_1 \in H^4(X;\mathbb{Q})$ can be written as $a\alpha + b\beta$ with $a, b \in \mathbb{Q}$, any Pontryagin class $p_2 \in H^8(X;\mathbb{Q})$ can be written as $c\alpha\beta$ with $c \in \mathbb{Q}$. To satisfy condition (i), we seek p_1, p_2 such that

$$L_2(p_1, p_2) = -\frac{1}{45}p_1^2 + \frac{7}{45}p_2 = (2ab + c)\alpha\beta = 0$$

There are obviously infinitely many triples (a, b, c) that satisfy this identity, which can realize infinitely many distinct pairs of Pontryagin numbers $p_{1,1} = 2ab$ and $p_2 = c$. For condition (ii), the equivariant intersection form $H^4(\widetilde{X};\mathbb{Q}) \times H^4(\widetilde{X};\mathbb{Q}) \to \mathbb{Q}$ defined as

$\langle \cdot \cup \cdot, x^3y \rangle$ is clearly hyperbolic. Then all these distinct Pontryagin numbers can be realized by smooth closed manifold with distinct homeomorphism types.

We have proved the following theorem:

THEOREM 5.2.7. *For any $\mathbb{Q}$-PDA $\mathcal{A}^*$ of the form in Case (i), which is isomorphic to $H^*(\mathbb{CP}^3 \times S^2; \mathbb{Q}) \cong \mathbb{Q}[x,y]/(y^2, x^4)$, there exists a free $\mathbb{Z}_2$-action on a smooth closed manifold whose rational cohomology ring realizes $\mathcal{A}^*$ with the action $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathcal{A}^2$. Such $\mathbb{Z}_2$-actions are the only possible finite action on $\mathcal{A}^*$ that can be realized by a free orientation-preserving action on a manifold whose ratioanl cohomology ring realizes $\mathcal{A}^*$. And the realizing manifolds fall into infinitely many homoemorphism types.*

Case (iii_1): $\mathcal{A}^* = \mathbb{Q}[x,y]/(kx^2 - y^2, x^4)$ with $k \neq 0$.

First notice that when $k \equiv 1 \pmod{\mathbb{Q}^{*2}}$, $\mathcal{A}^*$ is isomorphic to

$$H^*(\mathbb{CP}^4 \# \overline{\mathbb{CP}^4}; \mathbb{Q}) \cong \mathbb{Q}[x,y]/(xy, x^4 + y^4)$$

which was already discussed in Case (i). When $k \equiv -1 \pmod{\mathbb{Q}^{*2}}$, $\mathcal{A}^*$ is isomorphic to

$$\mathbb{Q}[x,y]/(x^2 + y^2, x^4)$$

When $k \not\equiv \pm 1 \pmod{\mathbb{Q}^{*2}}$, one can check that

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the only possible finite action on $\mathcal{A}^*$ with Lefschetz number zero. So $G = \mathbb{Z}_2$ is the only finite group that could possibly act freely on a manifold whose cohomology ring realizes $\mathcal{A}^*$.

When $k = -1$, i.e. the case $\mathcal{A}^* \cong \mathbb{Q}[x,y]/(x^2 + y^2, x^4)$

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are the only possible finite actions on $\mathcal{A}^*$ with Lefschetz number zero. So $G = \mathbb{Z}_2$, generated by $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $G = \mathbb{Z}_4$, generated by $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, are the only finite groups that could possibly act freely on a manifold whose cohomology ring realizes such $\mathcal{A}^*$.

We follow the same procedure as in the last two cases. Applying Theorem 5.1.1, all the above G-actions can be realized. We have proved the following theorem:

THEOREM 5.2.8. *For any $\mathbb{Q}$-PDA $\mathcal{A}^*$ of the form in Case (iii_1), there exists a free $\mathbb{Z}_2$-action on a smooth closed manifold whose rational cohomology ring realizes $\mathcal{A}^*$ with the action $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathcal{A}^2$. This is the only possible finite action on $\mathcal{A}^*$ with $k \not\equiv \pm 1 \pmod{\mathbb{Q}^{*2}}$ that can be realized by a free orientation-preserving action on a manifold whose rational cohomology ring realizes $\mathcal{A}^*$.*

THEOREM 5.2.9. *For $\mathbb{Q}$-PDA $\mathcal{A}^* = \mathbb{Q}[x,y]/(x^2+y^2, x^4)$, there exists a free $\mathbb{Z}_4$-action on a smooth closed manifold whose rational cohomology ring realizes $\mathcal{A}^*$ with the action $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathcal{A}^2$. This $\mathbb{Z}_4$-action together with the above $\mathbb{Z}_2$-action are the only possible finite actions on this $\mathcal{A}^*$ that can be realized by a free orientation-preserving action on a manifold whose rational cohomology ring realizes such $\mathcal{A}^*$.*

Case (iii_2): $\mathcal{A}^* = \mathbb{Q}[x,y]/(kx^2 - y^2, \ell x^4 - x^3 y)$ with $k \neq 0$.

When $k \equiv 1 \pmod{\mathbb{Q}^{*2}}$, any $\mathcal{A}^*$ in Case (iii_2) is isomorphic to a $\mathbb{Q}$-PDA of the form in Case (iii_1).

When $k \not\equiv \pm 1 \pmod{\mathbb{Q}^{*2}}$, $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathcal{A}^2$ is the only possible finite action on $\mathcal{A}^*$ with Lefschetz number zero. By condition (ii) in Theorem 5.1.1, this $\mathbb{Z}_2$-action can be realized by a free action on a smooth closed manifold whose cohomology ring is $\mathcal{A}^*$ if and only if

$$\ell^2 - k \equiv 1 \pmod{\mathbb{Q}^{*2}}$$

When $k \equiv -1 \pmod{\mathbb{Q}^{*2}}$, any $\mathcal{A}^*$ in Case (iii_2) is isomorphic to a $\mathbb{Q}$-PDA of the form $\mathbb{Q}[x,y]/(x^2+y^2, \ell x^4 - x^3 y)$. Then

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are the only possible finite action on $\mathcal{A}^*$ with Lefschetz number zero. By condition (ii) in Theorem 5.1.1, the corresponding $\mathbb{Z}_4$-action can be realized by a free action on a smooth closed manifold whose cohomology ring is $\mathcal{A}^*$ if and only if

$$\ell = 0$$

THEOREM 5.2.10. *For $\mathbb{Q}$-PDA $\mathcal{A}^* = \mathbb{Q}[x,y]/(x^2+y^2, x^3y)$, there exists a free $\mathbb{Z}_4$-action on a smooth closed manifold whose rational cohomology ring realizes $\mathcal{A}^*$ with the action $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathcal{A}^2$. This $\mathbb{Z}_4$ action is the only possible finite action on $\mathcal{A}^*$ that can be realized by a free orientation-preserving action on a manifold whose rational cohomology ring realizes $\mathcal{A}^*$.*

CHAPTER 6

Rational surgery with $\pi_1 = \mathbb{Z}$

We will study the rational surgery realization problem in dimensions $4k+1$ when $\pi_1 = \mathbb{Z}$. I wish to give an realization theorem on a general case when the starting local space X is a fibration over the circle. But at this point, I am only able to handle the case when $X = Y \times S^1$ where Y is a $4k$-dimensional $\mathbb{Q}$-local space. The following theorem says that X can be realized by a smooth closed manifold if and only if Y is realizable.

THEOREM 6.0.1. *Given a $4k+1$ dimensional $\mathbb{Q}$-local, $\mathbb{Q}$-Poincaré complex $X = Y \times S^1$ where Y is a $4k$-dimensional simply-connected $\mathbb{Q}$-local $\mathbb{Q}$-Poincaré complex, there exists a closed smooth $4k+1$ dimensional manifold M with $\pi_1(M) = \mathbb{Z}$, and a $\mathbb{Q}$-homotopy equivalence $f : M \longrightarrow X$ if and only if:*

There exist cohomology classes $p_i \in H^{4i}(Y;\mathbb{Q}), 1 \leq i \leq k$, and a fundamental class $\mu_Y \in H_{4k}(Y;\mathbb{Q}) \cong \mathbb{Q}$ such that

(i) $\langle L_k(p_1, \ldots, p_k), \mu_Y \rangle = \sigma(Y)$

(ii) *The intersection form on $H^{2k}(Y;\mathbb{Q})$ defined as $\langle \cdot \cup \cdot, \mu_Y \rangle$ is isomorphic to a form $m\langle 1 \rangle \oplus n\langle -1 \rangle$ for some integers m and n.*

(iii) *There exists a closed smooth $4k+1$ dimensional manifold M' and a map $\nu : M' \to S^1$ such that*

$$\langle p_I, \mu_Y \rangle = \langle p_I(\tau_{M'}) \cup \nu^* \alpha, [M'] \rangle$$

for all partitions I of k. Here $\alpha \in H^1(S^1)$ is a generator.

PROOF. ($\Longleftarrow$) If there exists a choice of cohomology classes $p_i \in H^*(Y;\mathbb{Q})$ and a fundamental class μ_Y satisfying the conditions (i),(ii) and (iii), by Theorem 3.1.1, there exists a manifold N and a rational homotopy equivalence $g : N \to Y$. Then obviously, $g \times Id : M = N \times S^1 \to M = Y \times S^1$ is a rational homotopy equivalence, so the rational homotopy type of X can be realized by the $4k+1$ dimensional closed smooth manifold $M = N \times S^1$.

($\Longrightarrow$): Suppose there exists a closed smooth $4k+1$ dimensional manifold M with $\pi_1(M) = \mathbb{Z}$, and a $\mathbb{Q}$-homotopy equivalence $f : M \longrightarrow X$. Then f induces an isomorphism on cohomology $f^* : H^*(X;\mathbb{Q}) \xrightarrow{\cong} H^*(M;\mathbb{Q})$, let $p_i = i^* p_{iX}$ where $f^* p_{iX} = p_i(\tau_M)$, $i = 1, \ldots, k$.

To prove condition (iii), let $\mu_Y \in H_{4k}(Y;\mathbb{Q})$ be the fundamental class such that $i_* \mu_Y = u^* \alpha \cap \mu$, where $\mu = f_*[M]$ and $u : X = Y \times S^1 \to S^1$ is the projection. Let $M' = M$ and $\nu = u \circ f$, then

$$\begin{aligned} \langle p_I, \mu_Y \rangle &= \langle p_{IX} \cup u^* \alpha, \mu \rangle \\ &= \langle p_I(\tau_M) \cup \nu^* \alpha, [M] \rangle \end{aligned}$$

For condition (i), notice that one can perturb the second coordinate of $f = (f_1, f_2) : M \to X = Y \times S^1$ so that $u \circ f : M \to S^1$ is transverse regular to a point $* \in S^1$, let $N^{4k} = (u \circ f)^{-1}(*)$ be the transverse inverse image. Let $i : N \hookrightarrow M$ be the inclusion, then

$$\begin{aligned} \langle p_I, \mu_Y \rangle &= \langle L_k(p_1, \ldots, p_k)) \cup u^* \alpha, \mu \rangle \\ &= \langle L_k(p_1, \ldots, p_k) \cup u^* \alpha, f_*[M] \rangle \\ &= \langle L_k(f^* p_1, \ldots, f^* p_k), f^* u^* \alpha \cap [M] \rangle \\ &= \langle L_k(p_1(\tau_M), \ldots, p_k(\tau_M)), i_*[N] \rangle \\ &= \langle L_k(p_1(\tau_N), \ldots, p_k(\tau_N)), [N] \rangle \\ &= \sigma(N) \\ &= \sigma(Y) \quad (*) \end{aligned}$$

The last equality $(*)$ is a consequence of the following Lemma 6.0.2, and condition (ii) also follows from Lemma 6.0.2.

LEMMA 6.0.2. *With the above settings, let*

$$\phi_N : H^{2k}(N;\mathbb{Q}) \times H^{2k}(N;\mathbb{Q}) \to \mathbb{Q}$$

be the symmetric forms defined as $\langle \cdot \cup \cdot, [N] \rangle$*, and let*

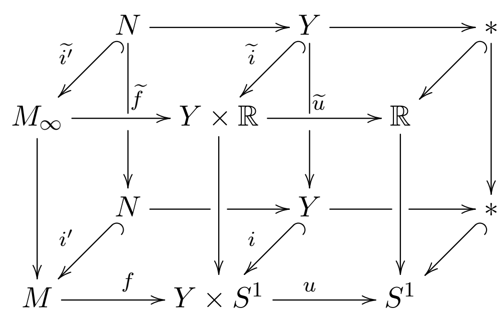
$$\phi_Y : H^{2k}(Y;\mathbb{Q}) \times H^{2k}(Y;\mathbb{Q}) \to \mathbb{Q}$$

be the symmetric forms defined as $\langle \cdot \cup \cdot, \mu_Y \rangle$*. Then in the Witt group,*

$$[\phi_N] = [\phi_Y]$$

.

Proof. We have the following commutative diagram:



where M_∞ is the infinite cyclic cover of M induced by the map $\exp : \mathbb{R} \to S^1$. As in the base space level, in the covering space level, we can perturb the second coordinate of $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) : M_\infty \to Y \times \mathbb{R}$ so that the map $\widetilde{u} \circ \widetilde{f}$ is transverse regular to the point $* = 0 \in \mathbb{R}$, then $\widehat{N} = (\widetilde{u} \circ \widetilde{f})^{-1}(*)$ is a $4k$ dimensional manifold that is homeomorphic to $N = (u \circ f)^{-1}(*)$, we will also use N to denote $\widehat{N}$. By construction, $\widetilde{f} : M_\infty \to Y \times \mathbb{R}$ is a proper rational homotopy equivalence.

As in the proof of Novikov conjecture for $\pi = \mathbb{Z}$ [**D2**], let:

$$\phi_{N \subset M_\infty} : H^{2k}(M_\infty;\mathbb{Q}) \times H^{2k}(M_\infty;\mathbb{Q}) \to \mathbb{Q}$$

be the symmetric bilinear form defined by $\phi_{N \subset M_\infty}(a', b') = \langle i'^* a' \cup i'^* b', [N] \rangle$ for $a', b' \in H^{2k}(M_\infty;\mathbb{Q})$. And let

$$\phi_{Y \subset Y \times \mathbb{R}} : H^{2k}(Y \times \mathbb{R};\mathbb{Q}) \times H^{2k}(Y \times \mathbb{R};\mathbb{Q}) \to \mathbb{Q}$$

be the form defined by $\phi_{Y \subset Y \times \mathbb{R}}(a, b) = \langle i^* a \cup i^* b, \mu_Y \rangle$ for $a, b \in H^{2k}(Y \times \mathbb{R};\mathbb{Q})$.

In the base space level, we have the identity

$$f_* i'_*[N] = f_*(f^* u^* \alpha \cap [M]) = u^* \alpha \cap f_*[M] = u^* \alpha \cap \mu = i_* \mu_Y$$

in $H_{4k}(X;\mathbb{Q}) = H_{4k}(Y \times S^1;\mathbb{Q})$. Since the map $H_*(Y \times \mathbb{R};\mathbb{Q}) \to H_*(Y \times S^1;\mathbb{Q})$ is injective, up in the covering space level, we have $\widetilde{f_*}\widetilde{i'}_*[N] = \widetilde{i}_* \mu_Y$, which implies that

$$\phi_{Y \subset Y \times \mathbb{R}} \cong \phi_{N \subset M_\infty}$$

It is clear that $\phi_Y \cong \phi_{Y \subset Y \times \mathbb{R}}$. Then to prove Lemma 6.0.2 which asserts $[\phi_N] = [\phi_Y]$, all we need to show is that $[\phi_N] = [\phi_{N \subset M_\infty}]$ in the Witt group. The following three lemmas will be used to prove this identity.

LEMMA 6.0.3. [**D2**, Lemma 2.5] *Let K be a compact set in X. Suppose X is filtered by*

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

Then there exist s an integer N so that for all $n \geq N$,

$$\phi_{K \subset X} \cong \phi_{K \subset X_n}$$

LEMMA 6.0.4. [**D2**, Lemma 2.6] *Let X^{4k+1} be a manifold with compact boundary ∂X. Then in the Witt group,*

$$[\phi_{\partial X}] = [\phi_{\partial X \subset X}]$$

PROOF. Let $L = \mathrm{Im}(i^* : H^{2k}(X, \mathbb{Q}) \to H^{2k}(\partial X;\mathbb{Q}))$. One can prove that $L^\perp \subset L$. Then

$$\begin{aligned} \phi_{\partial X} &\sim \phi_{\partial X}|_{H(L^\perp)^\perp} \\ &\sim \phi_{\partial X}|_{L/L^\perp} \\ &\sim \phi_{\partial X}|_{L} \\ &= \phi_{\partial X \subset X} \end{aligned}$$

□

LEMMA 6.0.5. *Suppose $\partial X \overset{i}{\hookrightarrow} X \overset{i'}{\hookrightarrow} X'$ where ∂X is the disjoint union $X_1 \coprod -X_2$. Then the symmetric forms*

$$\phi_{X_1 \subset X'} \cong \phi_{X_2 \subset X'}$$

PROOF. For any $a, b \in H^{2k}(X'; \mathbb{Q})$.

$$\begin{aligned}
\phi_{\partial X \subset X'}(a,b) &= \langle i^* i'^* a \cup i^* i'^* b, [\partial X] \rangle \\
&= \langle i^* i'^* a, i^* i'^* b \cap \partial_* [X] \rangle \\
&= \langle i^* i'^* a, \partial_* (i'^* b \cap [X] \rangle \\
&= \langle \delta^* i^* i'^* a, b \cap [X] \rangle \\
&= 0
\end{aligned}$$

This implies that $\phi_{(X_1 \coprod -X_2) \subset X'} = 0$, so $\phi_{X_1 \subset X'} = \phi_{X_2 \subset X'}$. □

Now we go back to the proof of Lemma 6.0.2. Let $M_{[x,y]} = (\widetilde{u} \circ \widetilde{f})^{-1}[x,y]$ for any subset $[x,y] \subset \mathbb{R}$, and $N_n = (\widetilde{u} \circ \widetilde{f})^{-1}(n)$ for any integer $n \in \mathbb{R}$. We have

$$\begin{aligned}
\phi_N &\cong \phi_{N_N} && (\text{for } N \gg 0, \text{ since } N_N \approx N) \\
&\sim \phi_{N_N \subset M_{[-\infty,N]}} && (\text{Lemma 6.0.3}) \\
&\cong \phi_{N \subset M_{[-\infty,N]}} && (\text{Lemma 6.0.5}, \quad \partial M_{[0,N]} = N_N \sqcup -N) \\
&\cong \phi_{N \subset M_\infty} && (\text{Lemma 6.0.4})
\end{aligned}$$

Then we have

$$\phi_N \sim \phi_{N \subset M_\infty} \cong \phi_{Y \subset Y \times \mathbb{R}} \cong \phi_Y$$

which completes the proof of Lemma 6.0.2 .

□

Back to the proof of the necessary direction of Theorem 6.0.1, Lemma 6.0.2 implies that the signature $\sigma(N) = \sigma(Y)$ and $\phi_Y \sim \phi_N = m\langle 1 \rangle \oplus n \langle -1 \rangle$, so conditions (i),(ii) and (iii) are all satisfied. By Theorem 3.1.1, there exists a choice of cohomology classes $p_i \in H^{4i}(Y; \mathbb{Q})$ and fundamental μ_Y if and only if Y can be realized by a $4k$ dimensional simply-connetced smooth closed manifold. So Theorem 6.0.1 implies that, for $X = Y \times S^1$, the rational homotopy type of X can be realized by a closed manifold if and only if Y is realizable. □

Bibliography

[AS] Atiyah, M.; Singer, I., *The index of elliptic operators. III.* Ann. of Math. (2) 87 1968 546–604.

[A] Anderson, G., *Surgery with coefficients* Lecture Notes in Mathematics, Vol. 591. Springer-Verlag, Berlin-New York, 1977.

[BK] Bousfield, A.; Kan, D., *Homotopy with respect to a ring.* Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), pp. 59–64. Amer. Math. Soc., Providence, R.I., 1971.

[B] Browder, W., *Surgery on simply-connected manifolds* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65. Springer-Verlag, New York-Heidelberg, 1972. ix+132 pp.

[DM] Davis, J.F.; Milgram, R.J., *Free actions on products of spheres: the rational case*, Math. Z. 207 (1991), no. 3, 379–390.

[D] Davis, J. F., *The rational symmetric signature of manifolds with finite fundamental group.* Topology 37 (1998), no. 4, 895–911.

[D2] Davis, J. F., *Manifold aspects of the Novikov conjecture.* Surveys on surgery theory. Vol. 1, 195–224, Ann. of Math. Stud., 145, Princeton Univ. Press, Princeton, NJ, 2000.

[FH] Félix, Y. ; Halperin, S., *Formal Spaces with Finite-Dimensional Rational Homotopy* Transactions of the American Mathematical Society, Vol. 270, No. 2 (Apr., 1982), pp. 575-588

[FHT] Félix, , Y. ; Halperin, S.; Thomas, J. *Rational homotopy theory.* (English summary) Graduate Texts in Mathematics, 205. Springer-Verlag, New York, 2001. xxxiv+535 pp. ISBN: 0-387-95068-0

[GM] Griffiths, P.; Morgan, J., *Rational homotopy theory and differential forms.* Progress in Mathematics, 16. Birkhuser, Boston, Mass., 1981.

[G] Gromov, M., *Positive curvature, macroscopic dimension, spectral gaps and higher signatures.* (English summary) Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993), 1–213, Progr. Math., 132, Birkhuser Boston, Boston, MA, 1996.

[L1] Little, R.D.,*The Pontrjagin class of a homotopy complex projective space*, Topology Appl. 34 (1990), no. 3, 257–267.

[L2] Little, R.D.,*Homotopy complex projective spaces with divisible splitting invariants*, Pacific J. Math. 140 (1989), no. 2, 251–262.

[MM] Madsen, I.; Milgram, J., *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, 92. Princeton University Press, 1971.

[MS] Milnor, J.W.; Stasheff, J.D., *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, 1974.

[M1] Milnor, J.,*Collected papers of John Milnor. III. Differential topology* American Mathematical Society, Providence, RI, 2007. xvi+343 pp. ISBN: 978-0-8218-4230-0; 0-8218-4230-7

[M2] Milnor, J. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963 vi+153 pp.

[PL] Papadima, S. ; Paunescu, L.,*Closed manifolds coming from Artinian complete intersections*, Trans. Amer. Math. Soc. 359 (2007), no. 6, 27772786

[P] Papadima, S., *Poincar duality algebras and the rational classification of differentiable manifolds.* Algebraic homotopy and local algebra (Luminy, 1982), 268–272, Astrisque, 113-114, Soc. Math. France, Paris, 1984.

[Q] Quinn, F., *Semifree group actions and surgery on PL homology manifolds*,Geometric topology (Proc. Conf., Park City, Utah, 1974), pp. 395–414. Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975.

[R] Ranicki, A.,*Algebraic and geometric surgery* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2002. xii+373 pp.

[S] Smale, S., *On the structure of manifolds.* Amer. J. Math. 84 1962 387–399.

[St1] Stong, R. E., *Notes on cobordism theory*, Mathematical notes Princeton University Press, 1968.

[St2] Stong, R. E., *Relations among characteristic numbers I.*,Topology 4 1965 267–281.

[S3] Sullivan, D., *Triangulating and smoothing homotopy equivalences and homeomorphisms* Geometric Topology Seminar Notes. The Hauptvermutung book, 69–103, K-Monogr. Math., 1, Kluwer Acad. Publ., Dordrecht, 1996.

[S2] Sullivan, D., *Geometric topology: localization, periodicity and Galois symmetry.* The 1970 MIT notes. Edited and with a preface by Andrew Ranicki. K-Monographs in Mathematics, 8. Springer, Dordrecht, 2005.

[S1] Sullivan, D., *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977), 269-331 (1978).

[TW] Taylor, L.; Williams, B., *Local surgery: foundations and applications*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), pp. 673–695, Lecture Notes in Math., 763, Springer, Berlin, 1979.

[W] Wall, C. T. C., *Determination of the Cobordism Ring.* The Annals of Mathematics, Second Series, Vol. 72, No. 2 (Sep., 1960), pp. 292-311